UNITÉ DE RECHERCHE
INRIA-RENNES

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tél.: (1) 39 63 55 11

Rapports de Recherche

N° 1226

Programme 1
Programmation, Calcul Symbolique et Intelligence Artificielle

CONDITIONAL REWRITE RULES AS AN ALGEBRAIC SEMANTICS OF PROCESSES

Eric BADOUEL

Mai 1990

RR_1226

IRISA

INSTITUT DE RECHERCHE EN INFORMATIQUE ET SYSTÈMES ALÉATOIRES

Campus Universitaire de Beaulieu
35042 - RENNES CÉDEX
FRANCE
Téléphone : 99 36 20 00
Télex : UNIRISA 950 473 F
Télécopie : 99 38 38 32

Conditional rewrite rules as an algebraic semantics of processes

Eric Badouel
Irisa, Campus de Beaulieu , 35042 Rennes Cedex

Publication Interne n° 528 - Mars 1990 - 46 Pages

Abstract : *We show that processes defined by equational identities together with conditional rewrite rules in Plotkin style can be given a purely algebraic description. A category of algebraic structures allowing for the specific nature of those languages is defined and its categorical properties are investigated. In particular it is shown how to derive new models from old by adding constraints either on the algebraic structures or on the categories in which their algebras lives.*

Sémantique algébrique des processus définis par réécritures conditionnelles

Résumé : *Nous montrons que les processus définis par la donnée mixte d'une présentation équationnelle et de règles de réécriture conditionnelles peuvent être décrits de façon purement algébrique. Une catégorie de structures algébriques prenant en compte la nature particulière de ces langages est définie et ses propriétés catégoriques sont étudiées. En particulier nous montrons qu'il est possible de dériver de nouveaux modèles de deux façons essentiellement indépendantes : on peut ajouter des contraintes sur les structures algébriques mises en jeu (e.g. en considérant des variétés équationnelles) mais également sur les catégories dans lesquelles les algèbres sont valuées.*

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE
(L. A. 227)
UNIVERSITÉ DE RENNES 1 I. N. S. A. DE RENNES

INSTITUT NATIONAL DE RECHERCHE
EN INFORMATIQUE ET EN AUTOMATIQUE
(LABORATOIRE DE RENNES)

Conditional rewrite rules as an algebraic semantics of processes

Eric Badouel
Irisa, Campus de Beaulieu , 35042 Rennes Cedex - France

Abstract

We show that processes defined by equational identities together with conditional rewrite rules in Plotkin style can be given a purely algebraic description. A category of algebraic structures allowing for the specific nature of those languages is defined and its categorical properties are investigated. In particular it is shown how to derive new models from old by adding constraints either on the algebraic structures or on the categories in which their algebras lives.

1 Introduction

A programming language can be construe as the algebraic structure of the closed rational trees corresponding to a finitary signature; a denotational semantics is then a structural translation of that language to an algebraic structure corresponding to a family of semantical domains (e.g. continuous lattices). This means we are given an interpretation of that programming language into a *generic* continuous lattice and therefore into any particular one. This is a general situation, whenever we have a translation $\alpha : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ of an algebraic structure into another one, we can deduce an interpretation of $\mathcal{S}_1$ into any particular model for $\mathcal{S}_2$: the meaning of an expression of $\mathcal{S}_1$ is given by the value of its α-translation in the particular model chosen for $\mathcal{S}_2$. We thus obtain at the semantical level, a correspondance in the opposite direction taking a model for $\mathcal{S}_2$ into a model for $\mathcal{S}_1$. This schema is a general feature of various functorial semantics known as the *Structure-Semantics duality.* This well-established duality enables one to devote most of one energy to the structural part and to get in a systematic way results for the corresponding models. Such a functorial semantics can therefore be reduced to the study of a category of algebraic structures and *relative interpretations* between them. A morphism $\alpha : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ between two algebraic structures is there called a relative interpretation of $\mathcal{S}_1$ into $\mathcal{S}_2$ since it corresponds to the models of $\mathcal{S}_1$ having (modulo α) a $\mathcal{S}_2$-shape. Since those relative interpretations may be composed they give rise to a hierarchy of more and more abstract meanings of a program.

We aim at giving such a functorial description for the behaviour of processes defined by conditional rewrite rules as advocated by Plotkin [Plo81]. That is we want to define a category of algebraic structures that allow for the specific nature of those languages. Let us recall the characteristic features of those process calculi. We first have a recursive term algebra that gives the syntax of the language. In order to avoid cumbersome considerations on syntax, we may suppose at this stage that some laws (as commutativity or associativity) hold for certains operators of the language. A program is then, up to those identities, a closed expression of the language. A set of conditional rewrite rules describes the behaviour of a program as the set of the elementary transitions it can perform. This allow us to define a *run* of a program as a sequence of such elementary transitions. From the runs of a program we can extract some properties we consider as relevant. A choice of such properties corresponds to an abstract view of a program behaviour and leads to an equivalence relation among the programs : two programs are equivalent when their runs satisfy the same properties. A process is then defined as a class of equivalent programs. We obtain a whole category of process calculi by considering various notions of observability for the program behaviour.

We shall now see how such a category of process calculi can be described in a purely algebraic manner. As it is well-known the syntax may be described as the algebraic structure of the rational terms built on the signature Σ which is made of the graded operators of the language. As those operator symbols are freely interpreted in that model, we shall call it the "Herbrand model". If equational identities are to be considered, we simply take the quotient of the Herbrand model corresponding to those equations. We still remain in the realms of (finitary) algebraic theory over sets that we shall describe as (finitary) monads over the category of sets. As for the operational models these are Σ-algebras (or (Σ, E)-algebras if equations are to be considered) equipped with a structure of transition systems that reflects the operational specifications. In order to characterize those operational models as algebraic structures we proceed in two stages. Firstly, we describe the algebraic structure of the proof trees for the deductif system which is made of the operational specifications. We then apply a "projection" that replaces each proof tree by the transition it proves. We obtain in this way, a monad defined on a category of transition systems whose algebras are exactly the operational models. This monad is a *lift* of the Herbrand model which means that if we forget about the additional structure of transition systems we recover the Herbrand model. This leads us to consider as algebraic structures those monads that are lifts of (finitary) algebraic structures over sets to categories of sets with some additional structures. We nevertheless shall make some completeness assumptions on those categories of sets with structure in order to ensure some completeness properties for the category of models itself.

Our approach is similar to a recent work by J. Meseguer [Mes90] in which we can find, as well, a mixed presentation of equational identities and rewrite rules. Nevertheless, though the notion of *concurrent rewriting* introduced by Meseguer proved to unify a wide variety of models for concurrent programs, it cannot take into account processes defined by conditional rewrite rules. This is so because the rewritings it considers are *context free* contrary to the conditional rewritings in Plotkin style for which the contexts play a central role. Actually this latter formalism merely consists in a definition of the transitions a context can perform in terms of the potential transitions of its components by inhibiting some of them and synchronizing some others. In this way those two notions of rewritings are complementary, and it is not obvious to unify them. Such a unification would amount to an axiomatisation of a general notion of rewriting within an (not necessarily free) algebra e.g. in the line of [Rao84]. The method advocated by Meseguer relies on the correspondance between deductive systems and categories introduced by Lambek [Lam69] (see also [Sza78]) and should be compared for instance to [Ben75], [Eyt80] and [RS87]. The context free assumption cannot be relaxed : for example in [RS87] where a 2-categorical treatment is given we see that it corresponds to an axiom of 2-categories. For that reason I had to proceed differently though the algebraic description of processes still arise quite naturally from the deductive system of the operational specifications.

The remaining of the paper is organized as follows. In section (2) we survey the results on monads that are necessary for the understanding of the remaining of the paper. In section (3) it is shown how the operational models can be described as algebras over a category of transitions systems. Our category of models is then introduced in section (4). It consists in a category of monads that are lifts of algebraic structure over sets to some categories of sets with structures. The completness property we assume for those categories of sets with structure ensure some completness property for the category of models. We use this property to show how quotients may be defined in that category. Section (5) is the conclusion.

2 Algebraic structures as monads

In this paragraph, we survey some results on monad theory (see e.g. [Man76] or [BW85] for a more detailed account) and show that this formalism is well suited for the description of algebraic structures appearing in computer science. As an introductory example let us consider a (finitary) signature i.e. a set Σ of operator symbols with arity $a : \Sigma \to \mathbf{N}$. We let Σ_n stands for the n-ary operator symbols of Σ. We recall that such a signature can be viewed as a set functor $\Sigma : \mathbf{Set} \to \mathbf{Set}$ defined on sets and mappings by

$$\Sigma A = \coprod_{f \in \Sigma} A^{a(f)} \quad \text{and} \quad \Sigma\varphi = \coprod_{f \in \Sigma} \varphi^{a(f)}$$

Since we have a distinguished coproduct structure on sets given by the disjoint unions, we can give a more concrete representation of ΣA as

$$\Sigma A = \coprod_{f\in\Sigma_n} A^n = \bigcup_{f\in\Sigma_n} \{f\} \times A^n = \{f[a_1,\ldots,a_n] : f \in \Sigma_n \text{ and } a_i \in A\}$$

And with that representation we have

$$\Sigma\varphi(f[a_1,\ldots,a_n]) = f[\varphi(a_1),\ldots,\varphi(a_n)]$$

for any mapping φ. Now a Σ-algebra is a pair (D,δ) in which D is non empty set (the *domain* of the algebra) together with a family $\delta = \{\delta_f \,; f \in \Sigma\}$ of mappings $\delta_f : D^{a(f)} \rightarrow D$ associated to each operator symbol of Σ. A morphism φ between two such algebras (D,δ) and (D',δ') is a mapping preserving the interpretations of each operator symbol $f \in \Sigma_n$ in both algebras :

$$\varphi(\delta_f(a_1,\ldots,a_n)) = \delta'_f(\varphi(a_1),\ldots,\varphi(a_n))$$

i.e. a Σ-algebra can be construe as a mapping

$$\delta = \coprod_{f\in\Sigma_n} \delta_f : \Sigma D \longrightarrow D$$

and an algebra morphism $\varphi : (A,a) \rightarrow (B,b)$ as a mapping between the underlying sets such that

$$\begin{tikzcd} \Sigma A \arrow[r, "a"] \arrow[d, "\Sigma\varphi"'] & A \arrow[d, "\varphi"] \\ \Sigma B \arrow[r, "b"] & B \end{tikzcd} \qquad\qquad \varphi \circ a = b \circ \Sigma\varphi$$

The free Σ-algebra corresponding to a set X of *generators* is obtained as the least set Y such that

$$X + \Sigma Y \subseteq Y$$

i.e. it is the least set, denoted $T(\Sigma, X)$, for which

1. $x \in X \implies <x> \in T(\Sigma, X)$
2. $f \in \Sigma_n$ and $t_1,\ldots,t_n \in T(\Sigma,X) \implies f[t_1,\ldots,t_n] \in T(\Sigma,X)$

where $<x>$ stands for the *atomic* term reduced to the generator (or variable) $x \in X$. We can extend the definition of $T(\Sigma,-)$ on mappings so as to obtain a functor (we shall write T for $T(\Sigma,-)$ as long as only one signature is concerned); for this we let, for any mapping $\varphi : X \rightarrow Y$, $T\varphi$ be inductively defined by :

1. $T\varphi(<x>) = <\varphi(x)>$

2. $T\varphi(f[t_1,\dots,t_n]) = f[T\varphi(t_1),\dots,T\varphi(t_n)]$

which means that to apply $T\varphi$ to a term amounts to apply φ to each of its variables. If (D,δ) is any Σ-algebra, we notice that a term $t \in TD$ is a *syntactical expression* that refers to a specific element of the domain D; the mapping $\delta^* : TD \to D$ that takes such a term to its corresponding value in D is inductively defined by :

1. $\delta^*(<x>) = x$ and,

2. $\forall f \in \Sigma_n \quad \delta^*(f[t_1,\dots,t_n]) = \delta(f[\delta^*(t_1),\dots,\delta^*(t_n)])$

Those two equations means respectively that this evaluation mapping respects the atomic structures and the substitution operation. More precisely, we define an *embedding of generators* as the mapping $\eta_D : D \to TD$ that takes each generator $x \in D$ to the corresponding atomic term $<x> \in TD$ and a *substitution operation* $\mu_D : TTD \to TD$ through the following statements

$$\begin{aligned}\mu_D(<t>) &= t \\ \mu_D(f[t_1,\dots,t_n]) &= f[\mu_D(t_1),\dots,\mu_D(t_n)]\end{aligned}$$

and the two previous equations are easily seen equivalent to

1. $\delta^* \circ \eta_D = 1_D$ and,

2. $\delta^* \circ \mu_D = \delta^* \circ T\delta^*$ respectively;

which says that the evaluation commutes with the substitution and that the evaluation of an atomic structure gives the corresponding generator. Such a triple $\mathbf{T} = (T,\eta,\mu)$ is a *monad* and a mapping $\alpha : TD \to D$ that satisfy both previous equations is said to be an algebra for that monad. We see that a $\mathbf{T}$-algebra for the preceding monad of terms is the same data as a Σ-algebra : the elements of ΣD can be identified with the elements of TD having the form $f[<x_1>,\dots,<x_n>]$ for $f \in \Sigma_n$ and $x_i \in D$ and therefore any $\mathbf{T}$-algebra (D,δ) gives rise, by restriction, to a Σ-algebra $\Sigma D \to TD \xrightarrow{\delta} D$ whose inductive extension is none other than δ. Moreover this correspondance can be extended to an isomorphism of categories povided that the $\mathbf{T}$-algebras morphisms from (D,δ) to (D',δ') are defined as those mappings $\varphi : D \to D'$ for which $\varphi \circ \delta = \delta' \circ T\varphi$. Now we give the general definitions :

Definition 1 *A* **monad** *on a category $\mathcal{C}$ is a triple (T,η,μ) in which T is an endofunctor of $\mathcal{C}$ and $\eta : I \dot{\to} T$ and $\mu : T^2 \dot{\to} T$ are both natural transformations called respectively* **the embedding of the generators** *and the* **substitution operation** *making the following diagrams commute*

$$
\begin{tikzcd}
TTT \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & TT \arrow[d, "\mu"] \\
TT \arrow[r, "\mu"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\eta T"] \arrow[dr, "1_T"'] & TT \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "1_T"] \\
& T &
\end{tikzcd}
$$

Those commuting diagrams state that the substitution operation is associative and that the embedding of the generators is neutral with regard to this substitution operation. A $\mathbf{T}$*-algebra is a pair* (A, a) *that consists in a* $\mathcal{C}$*-object* A *and an arrow* $a : TA \to A$ *called respectively the* **domain** *and* **structure map** *of the algebra making the following diagram commute*

$$
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "a"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
TTA \arrow[r, "\mu_A"] \arrow[d, "Ta"'] & TA \arrow[d, "a"] \\
TA \arrow[r, "a"] & A
\end{tikzcd}
$$

Those commuting diagrams state that the value of an atomic structure is given by the corresponding generator and that the evaluation commutes with the substitution operation. A $\mathbf{T}$*-algebra morphism from* (A, a) *to* (B, b) *is any arrow* $\varphi : A \to B$ *in* $\mathcal{C}$ *such that*

$$
\begin{tikzcd}
TA \arrow[r, "a"] \arrow[d, "T\varphi"'] & A \arrow[d, "\varphi"] \\
TB \arrow[r, "b"] & B
\end{tikzcd}
\qquad\qquad
\varphi \circ a = b \circ T\varphi
$$

The $\mathbf{T}$*-algebras and their morphisms form a category* $\mathcal{C}^{\mathbf{T}}$ *called the Eilenberg-Moore category for the monad* $\mathbf{T}$*. This category comes equipped with a forgetful functor* $U^{\mathbf{T}} : \mathcal{C}^{\mathbf{T}} \to \mathcal{C}$ *that takes an algebra* (A, a) *to its underlying domain* A *and a morphism* φ *to itself (considered as an arrow in* $\mathcal{C}$*).*

Let us give a few example of monads which are not used in the sequel of the paper but illustrate the preceding definitions.

Example 2 The monad of lists

It consists in the triple (L, η, μ) *in which the list constructor* L *takes a set* X *to the free monoid* X^* *and a mapping* $\varphi : X \to Y$ *to its alphabetic extension* $\varphi^* : X^* \to Y^*$ [1]

$$\varphi^*(\epsilon) = \epsilon \quad \textit{and} \quad \varphi^*(<x> \bullet m) = <\varphi(x)> \bullet \varphi^*(m)$$

[1] ϵ stands for the empty word and $\bullet$ for the concatenation in both monoids; x is an element of X and m an element of X^*.

The embedding of generators $\eta_X : X \to X^*$ takes each variable $x \in X$ to the word of length 1 reduced to that letter; and the substitution operation $\mu_X : (X^*)^* \to X^*$ takes a list of words $< m_1, \ldots, m_n >$ of X^* to their concatenation $m_1 \bullet \ldots \bullet m_n$.
An $\mathbf{L}$-algebra is equivalent to a monoid. More precisely if (M, π) is an $\mathbf{L}$-algebra, $\pi : M^* \to M$ appears as some generalized product *on M and we can associate to it the monoid $(M, e, \circ)$ given by $e = \pi(\epsilon)$ and $x \circ y = \pi(<x, y>)$. We check that the associativity of $\circ$ and the fact that e is a neutral element follows from the axioms for $\mathbf{L}$-algebras. Conversely, we can turn any monoid $(M, e, \circ)$ into an $\mathbf{L}$-algebra (M, π) with $\pi(\epsilon) = e$ and $\pi(x \bullet m) = x \circ \pi(m)$. We check that π is indeed an $\mathbf{L}$-algebra structure on M and that both correspondance are inverse to each other.*

Example 3 The monad of the right actions of a monoid
Let $\mathcal{M} = (M, \bullet, e)$ be a monoid, we consider the monad $\mathbf{T} = (T, \eta, \mu)$ on Set for which the functor T is defined on sets and mappings by $TX = X \times M$ and $Tf = f \times 1_M$ respectively and the natural transformation η and μ are given by

$$\begin{array}{llll} \eta_X & : X \to X \times M & : & x \mapsto <x, e> \\ \mu_X & : (X \times M) \times M \to X \times M & : & <<x, m>, n> \mapsto <x, m \bullet n> \end{array}$$

A $\mathbf{T}$-algebra is a deterministic transition system with transitions labelled in A. Actually, it is a set Q of states together with a transition mapping $\delta : Q \times M \to Q$ making the following diagrams commute

$$\begin{tikzcd} Q \arrow[r, "\eta_Q"] \arrow[dr, "1_Q"'] & Q \times M \arrow[d, "\delta"] \\ & Q \end{tikzcd} \qquad \begin{tikzcd} (Q \times M) \times M \arrow[r, "\mu_Q"] \arrow[d, "\delta \times 1_M"'] & Q \times M \arrow[d, "\delta"] \\ Q \times M \arrow[r, "\delta"] & Q \end{tikzcd}$$

If we let $x \xrightarrow{m} y \in \delta$ stands for $\delta(x, m) = y$ the two previous commuting diagrams correspond to the equations

1. $x \xrightarrow{\epsilon} x \in \delta$ *and*
2. $x \xrightarrow{m} y \in \delta$ *and* $y \xrightarrow{n} z \in \delta \implies x \xrightarrow{m \bullet n} z \in \delta$

A morphism between two such transition systems is a mapping $f : Q \to Q'$ between their respective sets of states such that

$$\begin{tikzcd} Q \times M \arrow[r, "\delta"] \arrow[d, "f \times M"'] & Q \arrow[d, "f"] \\ Q' \times M \arrow[r, "\delta'"] & Q' \end{tikzcd} \qquad x \xrightarrow{m} y \in \delta \Rightarrow f(x) \xrightarrow{m} f(y) \in \delta'$$

Example 4 The powerset monads
Any adjunction situation $\eta, \epsilon : F \dashv U : \mathcal{X} \longrightarrow \mathcal{C}$ gives rise to a monad $\mathbf{T} = (T, \eta, \mu)$ on $\mathcal{C}$ with $T = UF$, $\eta : I \rightarrow UF$ be the unit of the adjunction and $\mu = U\epsilon F$ where $\epsilon : FU \rightarrow I$ is the co-unit of the adjunction. For instance the algebra of terms we gave as an introductory example corresponds to the adjunction situation $\eta, \epsilon : F \dashv U : \Sigma\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}$ in which U is the forgetful functor and F corresponds to the definition of the free Σ-algebras. For another example, let X be a set and $P(X) = \{Y \; ; \; Y \subseteq X\}$ be the powerset of X equipped with its natural structure of upper semi lattice. We extend P into a functor by letting $P\varphi$ be the additive extension of φ. In this way we obtain a functor $P : \mathbf{Ens} \rightarrow \mathbf{USL}$ from the category of sets to the category of upper semi lattices and continuous mappings. We check that P is left adjoint to the forgetful functor and that the resulting monad $(\mathcal{P},\eta,\mu)$ is given by : $\mathcal{P} = UP$; $\eta_X(x) = \{x\}$ and $\mu_X(e) = \bigcup_{E \in e} E$
For α a regular cardinal, we get an analogous monad $\mathcal{P}_\alpha$ by restricting oneself to the subset of cardinality less than α (for instance the finite sets if $\alpha = \omega$).
A $\mathcal{P}_\omega$-algebra $(A, \vee)$ is a semi-lattice :

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & \mathcal{P}_\omega A \arrow[d, "\vee"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}_\omega\mathcal{P}_\omega A \arrow[r, "\mu_A"] \arrow[d, "\mathcal{P}_\omega\vee"'] & \mathcal{P}_\omega A \arrow[d, "\vee"] \\ \mathcal{P}_\omega A \arrow[r, "\vee"] & A \end{tikzcd}$$

$\vee\{a\} = a$ *associativity of* $\vee$

and their morphisms are monotonic mappings. As for $\mathcal{P}$ we get the complete upper semi-lattices and the continuous mappings.

The following result state the existence of free $\mathbf{T}$-algebras.

Proposition 5 *The forgetful functor $U^{\mathbf{T}} : \mathcal{C}^{\mathbf{T}} \rightarrow \mathcal{C}$ has a left adjoint $F^{\mathbf{T}}$ given by $F^{\mathbf{T}}A = (TA, \mu_A)$ on objects and $F^{\mathbf{T}}\varphi = T\varphi$ on arrows. The unit of the adjunction is η, and the component of the co-unit $\epsilon^{\mathbf{T}}$ at a $\mathbf{T}$-algebra (A, a) is $a : (TA, \mu_A) \rightarrow (A, a)$ which indeed is a $\mathbf{T}$-algebra morphism. Moreover we have $U^{\mathbf{T}}\epsilon^{\mathbf{T}}F^{\mathbf{T}} = \mu$ i.e. the monad that results from that adjunction is none other than the monad $\mathbf{T}$ we started from.*

Let V be an object of $\mathcal{C}$, (TV, μ_V) is called the *free* $\mathbf{T}$*-algebra over* V. Thanks to the previous proposition it actually verify the following universal property : given a *domain* D, a *valuation* $v : V \rightarrow D$ and an *interpretation* (i.e. a $\mathbf{T}$-algebra structure δ on D) there exists a unique arrow v_δ^* such that both following diagrams commute

$$
\begin{tikzcd}
V \arrow[r, "\eta_V"] \arrow[dr, "v"'] & TV \arrow[d, "v^*_\delta"] & TTV \arrow[r, "\mu_V"] \arrow[d, "Tv^*_\delta"'] & TV \arrow[d, "v^*_\delta"] \\
 & D & TD \arrow[r, "\delta"] & D
\end{tikzcd}
$$

When the base category $\mathcal{C}$ is the category of sets, for any $t \in TV$, $v^*_\delta(t)$ gives the value of t in the *valued interpretation* $(D, \delta; v)$. In the general case the following identity is readily verified

$$v^*_\delta = \delta \circ Tv : TV \xrightarrow{Tv} TD \xrightarrow{\delta} D$$

An important issue in monad theory is to know whether a given category $\mathcal{X}$ can be viewed as some category of algebras. This is always defined in relation to some "forgetful functor" : a functor $U : \mathcal{X} \to \mathcal{C}$ is said to be monadic (resp. weakly monadic) if it has a left adjoint F (i.e. free structures exists) and if moreover $\mathcal{X}$ is isomorphic (resp. equivalent) to the Eilenberg-Moore category for the monad resulting from that adjunction.

A monad describe an algebraic structure. A monad morphism corresponds to some translation of an algebraic structure into another one; such a translation must preserve the atomic structures and the substitution operations :

Definition 6 *Let* $\mathbf{T} = (T, \eta, \mu)$ *and* $\mathbf{T}' = (T', \eta', \mu')$ *be both monads on a same category* $\mathcal{C}$*, a morphism from* $\mathbf{T}$ *into* $\mathbf{T}'$ *is a natural transformation* $\alpha : T \dot{\to} T'$ *making the following diagrams commute.*

$$
\begin{tikzcd}
I \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & T \arrow[d, "\alpha"] & TT \arrow[r, "\mu"] \arrow[d, "\alpha\alpha"'] & T \arrow[d, "\alpha"] \\
 & T' & T'T' \arrow[r, "\mu'"] & T'
\end{tikzcd}
$$

where $\alpha\alpha = \alpha T' \circ T\alpha = T'\alpha \circ \alpha T$ *is the horizontal composition of natural transformations.*

We therefore obtain a category $\mathcal{M}on(\mathcal{C})$ of monads corresponding to a fixed base category $\mathcal{C}$ for which we can describe a structure-semantics duality. For one hand if $\alpha : \mathbf{T} \to \mathbf{S}$ is any monad morphism in $\mathcal{M}on(\mathcal{C})$, we obtain by *composition with* α a correspondance between the category of algebras in the opposite direction :

$$\mathcal{C}^\alpha : \mathcal{C}^{\mathbf{S}} \longrightarrow \mathcal{C}^{\mathbf{T}} \;=\; \begin{cases} (A, a) & \mapsto \quad (A, TA \xrightarrow{\alpha_A} SA \xrightarrow{a} A) \\ \quad f & \mapsto \quad f \end{cases}$$

and moreover this functor respects the underlying sets, i.e. commutes with the forgetful functors. Conversely, if $U : \mathcal{C}^{\mathbf{S}} \to \mathcal{C}^{\mathbf{T}}$ commutes with the forgetful functors, we let σ_A be the $\mathbf{T}$-algebra structure on SA corresponding to the free algebra $(SA, \mu^{\mathbf{S}}_A)$; and the family of arrows

$$\{\alpha_A = TA \xrightarrow{T\eta^{\mathbf{S}}_A} TSA \xrightarrow{\sigma_A} SA \quad ; \quad A \in |\mathcal{C}|\}$$

proves to be a monad morphism from $\mathbf{T}$ to $\mathbf{S}$. And this correspondance is the inverse of the correspondance taking α to $\mathcal{C}^{\alpha}$. To end with, we just mention a few facts that concern the changing of the base category. If $\mathbf{T}$ and $\mathbf{S}$ are monads on $\mathcal{C}$ and $\mathcal{D}$ respectively, a **generalized monad morphism** $(U, \alpha) : (\mathcal{C}, \mathbf{T}) \to (\mathcal{D}, \mathbf{S})$ consists in a functor U from $\mathcal{C}$ to $\mathcal{D}$ together with a natural transformation

$$\begin{tikzcd} \mathcal{C} \arrow[r, "U"] \arrow[d, "T"'] & \mathcal{D} \arrow[d, "S"] \arrow[dl, phantom, "\alpha\ \swarrow"] \\ \mathcal{C} \arrow[r, "U"'] & \mathcal{D} \end{tikzcd} \qquad\qquad \alpha : S \circ U \longrightarrow U \circ T$$

for which both following diagrams commute

$$\begin{tikzcd} & SU \arrow[dd, "\alpha"] \\ U \arrow[ur, "\eta^{\mathbf{S}}U"] \arrow[dr, "U\eta^{\mathbf{T}}"'] & \\ & UT \end{tikzcd} \qquad \begin{tikzcd} SSU \arrow[r, "S\alpha"] \arrow[d, "\mu^{\mathbf{S}}U"'] & SUT \arrow[r, "\alpha T"] & UTT \arrow[d, "U\mu^{\mathbf{T}}"] \\ SU \arrow[rr, "\alpha"] & & UT \end{tikzcd}$$

The composition of such morphisms $(\mathcal{C}, \mathbf{T}) \xrightarrow{(U,\alpha)} (\mathcal{D}, \mathbf{S}) \xrightarrow{(V,\beta)} (\mathcal{E}, \mathbf{R})$ is given by $(V \circ U, V\alpha \circ \beta U) : (\mathcal{C}, \mathbf{T}) \longrightarrow (\mathcal{E}, \mathbf{R})$. Of course, the category $\mathcal{M}on(\mathcal{C})$ (more precisely its dual) is recovered as the fibre corresponding to $\mathcal{C}$, i.e. by restricting oneself to the morphisms having the form $(1_{\mathcal{C}}, \alpha) : (\mathcal{C}, \mathbf{T}) \to (\mathcal{C}, \mathbf{T}')$. More details are to be found in [Str72], we just mention here that there exists a bijective correspondance between the natural transformations $\alpha : SU \to UT$ for which (U, α) is a generalized monad morphism from $(\mathcal{C}, \mathbf{T})$ to $(\mathcal{D}, \mathbf{S})$ and the *lifts* of U to the categories of algebras these are functors $U^*_\alpha : \mathcal{C}^{\mathbf{T}} \to \mathcal{D}^{\mathbf{S}}$ such that

$$\begin{tikzcd} \mathcal{C}^{\mathbf{T}} \arrow[r, "U^*_\alpha"] \arrow[d, "U^{\mathbf{T}}"'] & \mathcal{D}^{\mathbf{S}} \arrow[d, "U^{\mathbf{S}}"] \\ \mathcal{C} \arrow[r, "U"'] & \mathcal{D} \end{tikzcd} \qquad\qquad U^{\mathbf{S}} \circ U^*_\alpha = U \circ U^{\mathbf{T}}$$

Note : if $U = 1_{\mathcal{C}}$ (hence $\mathcal{C} = \mathcal{D}$) and $\alpha : \mathbf{S} \to \mathbf{T}$ is a monad morphism, then $(1_{\mathcal{C}})^*_\alpha$ is none other than the functor $\mathcal{C}^{\alpha}$ previously defined.

3 The operational model

In this section we introduce the operational models corresponding to a set of conditional rewrite rules in Plotkin style and show that these models can be characterized as the algebras for a monad over a category of transition systems.

3.1 Structural operational specifications

The behaviour of programs is given by a set of structural rules describing the elementary transitions a program can perform. As an example, let us consider the restriction of CCS [Mil80] defined as follows. We have a set $A = \Lambda \cup \{\tau\}$ of action names where τ stands for a so called *invisible action* and Λ is equipped with an involutive mapping of synchronization $\overline{()} : \Lambda \longrightarrow \Lambda$; the complementary actions a and $\bar{a}$ are those taking place in a communication (delivering and reception of a message). The language we consider is given by the signature $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ that consists in a constant standing for the inaction $\Sigma_0 = \{nil\}$, together with unary operators of prefixing and restriction for each action in Λ

$$\Sigma_1 = \{a.\text{-}\,;\ a \in \Lambda\} \cup \{(\text{-})\backslash_a\,;\ a \in \Lambda\}$$

and two binary operators corresponding to the non deterministic choice and the parallel composition : $\Sigma_2 = \{+; \|\}$. A program is identified with a closed rational tree built over that signature and its behaviour is given through the following structural rules.

$$\begin{array}{rcl}
\ldots & \Longrightarrow & a.t \xrightarrow{a} t \\
t \xrightarrow{a} t' & \Longrightarrow & t + u \xrightarrow{a} t' \\
u \xrightarrow{a} u' & \Longrightarrow & t + u \xrightarrow{a} u' \\
t \xrightarrow{a} t' & \Longrightarrow & t\|u \xrightarrow{a} t'\|u \\
u \xrightarrow{a} u' & \Longrightarrow & t\|u \xrightarrow{a} t\|u' \\
t \xrightarrow{a} t' \quad et \quad u \xrightarrow{\bar{a}} u' & \Longrightarrow & t\|u \xrightarrow{\tau} t'\|u' \\
t \xrightarrow{b} t' \quad et \quad b \neq a & \Longrightarrow & t\backslash_a \xrightarrow{b} t'\backslash_a
\end{array}$$

An *operational model* (X, α, ξ) is a set X equipped with a $\mathbf{T}$-algebra structure $\alpha : TX \longrightarrow X$ and a structure of transition systems $\xi \subset X \times A \times X$ satisfying the following conditions

$$\begin{array}{rcl}
\ldots & \Longrightarrow & \alpha(a.x) \xrightarrow{a} x \in \xi \\
x \xrightarrow{a} x' \in \xi & \Longrightarrow & \alpha(x + y) \xrightarrow{a} x' \in \xi \\
y \xrightarrow{a} y' \in \xi & \Longrightarrow & \alpha(x + y) \xrightarrow{a} y' \in \xi \\
x \xrightarrow{a} x' \in \xi & \Longrightarrow & \alpha(x|y) \xrightarrow{a} \alpha(x'|y) \in \xi \\
y \xrightarrow{a} y' \in \xi & \Longrightarrow & \alpha(x|y) \xrightarrow{a} \alpha(x|y') \in \xi \\
x \xrightarrow{a} x' \in \xi \quad et \quad y \xrightarrow{\bar{a}} y' \in \xi & \Longrightarrow & \alpha(x|y) \xrightarrow{\tau} \alpha(x'|y') \in \xi \\
x \xrightarrow{b} x' \in \xi \quad et \quad b \neq a & \Longrightarrow & \alpha(x\backslash_a) \xrightarrow{b} \alpha(x'\backslash_a) \in \xi
\end{array}$$

which means that each operational rule is satisfied when we interpret it in the triple (X, α, ξ). Such a triple will be called a *premodel*. More generally we shall consider the format of rules introduced by Robert de Simone [dS84] : for each symbol operator $f \in \Sigma_n$ we have a set R_f of rules schema having the form

$$(p_i \xrightarrow{b_i} p_i' \, ; \, i \in I_r) \quad \Longrightarrow \quad f[p_0, \ldots, p_{n-1}] \xrightarrow{a} C_r[q_0, \ldots, q_{n-1}] \quad if \;\; \phi_r(b, a)$$
$$where \;\; q_j = p_j \;\; if \;\; j \notin I_r \;\; and \;\; q_j = p_j' \;\; if \;\; j \in I_r$$

such a rule schema corresponds to a triple $r = < I_r, \phi_r, C_r >$ where

- $I_r \subset \{0, \ldots, n-1\}$ gives the components p_i involved by the rule,
- ϕ_r is a relation between the actions of the components and the resulting action for $f[p_0, \ldots, p_{n-1}]$ and,
- C_r is a context with holes labelled in $n = \{0, \ldots, n-1\}$ i.e. it is an element of Tn where T is the term functor for the signature Σ.

R$= \cup_{f \in \Sigma} R_f$ stands for the whole set of rules.

Let (X, α, ξ) be a premodel. Since each term $t \in TX$ refers, thanks to α, to a specific element of X; each transition between two such terms stands for a transition between element of X. If this transition is in ξ we say that (X, α, ξ) *validates* the formal transition between terms. That is we define a validation relation by

$$(X, \alpha, \xi) \models t \xrightarrow{a} t' \quad \Longleftrightarrow \quad \alpha(t) \xrightarrow{a} \alpha(t') \in \xi$$

An operational model is then a premodel that validates the conclusion of a rule whenever its premisses are fulfilled.

Definition 7 *A premodel (X, α, ξ) is an operational model if for each intanciation of a rule $r \in R$:*

$$(x_i \xrightarrow{b_i} x_i' \, ; \, i \in I_r) \quad \Longrightarrow \quad f[x_0, \ldots, x_{n-1}] \xrightarrow{a} C_r[y_0, \ldots, y_{n-1}] \quad if \;\; \phi_r(b, a)$$
$$where \;\; y_j = x_j \;\; if \;\; j \notin I_r \;\; and \;\; y_j = x_j' \;\; if \;\; j \in I_r$$

we have $(X, \alpha, \xi) \models f[x_0, \ldots, x_{n-1}] \xrightarrow{a} C_r[y_0, \ldots, y_{n-1}]$ whenever $x_i \xrightarrow{b_i} x_i' \in \xi$ for $i \in I_r$ and $\phi_r(b, a)$.

We write **SOS** in short for **S**tructural **O**perational **S**pecification to stand for the deductive system corresponding to the operational rules. We can prove by induction on the structure of the proofs that the operational models are the premodels that validates the provable formal transitions on terms :

Proposition 8 *A premodel (X, α, ξ) is an operational model if, and only if, for any formal transition $t \xrightarrow{a} t' \in TX \times A \times TX$ one has*

$$\mathbf{SOS}, \xi \vdash t \xrightarrow{a} t' \quad \Longrightarrow \quad (X, \alpha, \xi) \models t \xrightarrow{a} t' \tag{1}$$

where $\mathbf{SOS}, \xi \vdash t \xrightarrow{a} t'$ means that the transition $t \xrightarrow{a} t' \in TX \times A \times TX$ is provable in the deductive system obtained by adding to **SOS** *the transitions of ξ as additional axioms (more precisely we add the axiom $< x > \xrightarrow{a} < x' >$ for each transition $x \xrightarrow{a} x'$ in ξ).*

For instance the premodel in which we could find three elements x, x' and y such that $x \xrightarrow{a} x' \in \xi$ and $\alpha(x + y) \xrightarrow{a} x' \notin \xi$ is rejected as an operational model since the transition $x + y \xrightarrow{a} x'$ is provable

$$\mathbf{SOS}, x \xrightarrow{a} x' \vdash x + y \xrightarrow{a} x'$$

with the valid (w.r.t. ξ) assumption $x \xrightarrow{a} x'$ and still is not valid itself :

$$(X, \alpha, \xi) \not\models x + y \xrightarrow{a} x'$$

In the remaining of this section we show that the operational models are the algebras for a monad over a category of transition systems.

3.2 The proof trees

This fact will easily follows from a few elementary properties of the deductive system **SOS** we shall now survey. We mainly show that **SOS** corresponds to an algebraic structure $\mathcal{D}$ (for $\mathcal{D}$eduction) on a category $\mathcal{K}$ whose objects (X, α, Γ) are sets equipped with a **T**-algebra structure α and a *set of assumptions* on X which is a partial mapping

$$\Gamma : \mathcal{N} \longrightarrow X \times A \times X$$

taking some *transition names* to actual transitions between elements of X. A morphism $< f, \varphi >: (X, \alpha, \Gamma) \longrightarrow (Y, \beta, \Delta)$ in that category is a morphism of **T**-algebras $\varphi : (X, \alpha) \to (Y, \beta)$ together with a mapping $f : \mathcal{N}_\Gamma \to \mathcal{N}_\Delta$ such that

$$(\varphi \times 1_A \times \varphi) \circ \Gamma = \Delta \circ f$$

$$\begin{tikzcd} \mathcal{N}_\Gamma \arrow[r, "\Gamma"] \arrow[d, "f"'] & X \times A \times X \arrow[d, "\varphi \times 1_A \times \varphi"] \\ \mathcal{N}_\Delta \arrow[r, "\Delta"] & Y \times A \times Y \end{tikzcd} \qquad \begin{tikzcd} (X, \alpha) \arrow[d, "\varphi"] \\ (Y, \beta) \end{tikzcd}$$

Let (X, α, Γ) be an object in $\mathcal{K}$, the proof of a transition between elements of X under assumptions Γ can be described as a proof tree. The set of such proof trees $\mathbf{AR\text{-}P}(\mathcal{N}, X)$ is the set of terms $T(\Sigma_X, \mathcal{N})$ corresponding to the signature Σ_X given by

1. each element in X is a null-ary operator for Σ_X,
2. if $r = < I_r, \phi_r, C_r >$ is a rule schema corresponding to an operator symbol $f \in \Sigma_n$ and if a is an arbitrary action, then $(r : a)$ is an n-ary constructor of proof trees.

and whose variables $\mathcal{N}$ corresponds to *transition names*. Consequently, for each set X we have a monad $(\mathbf{AR\text{-}P}(-, X), \mathfrak{e}_{-,X}, \mathfrak{m}_{-,X})$ whose embedding of generators

$$\mathfrak{e}_{\mathcal{N},X} : \mathcal{N} \longrightarrow \mathbf{AR\text{-}P}(\mathcal{N}, X)$$

takes a transition name (also called *assumption*) to the elementary proof reduced to that assumption, and whose substitution operation

$$\mathfrak{m}_{\mathcal{N},X} : \mathbf{AR\text{-}P}(\mathbf{AR\text{-}P}(\mathcal{N}, X), X) \longrightarrow \mathbf{AR\text{-}P}(\mathcal{N}, X)$$

takes a proof tree A whose assumptions are themselves proof trees $A_1, \ldots, A_n$ into the proof tree obtained by replacing in A each occurrence of a proof tree A_i by its value.

Of course most proof trees do actually proves no transitions. For every $\mathbf{T}$-algebra (X, α) and set of assumptions $\Gamma : \mathcal{N} \to X \times A \times X$, we define a partial mapping

$$\mathcal{D}_{\Gamma,\alpha} : \mathbf{AR\text{-}P}(\mathcal{N}, X) \longrightarrow X \times A \times X$$

that takes each proof tree to the transition it proves (if any) [2]. $\mathcal{D}_{\Gamma,\alpha}$ is defined inductively as follows :

1. $\mathcal{D}_{\Gamma,\alpha}(x) = \perp$ for $x \in X$, i.e. an element of X proves no transition,
2. $\mathcal{D}_{\Gamma,\alpha}(<n>) = \Gamma_n$ for $n \in \mathcal{N}$,
3. $\mathcal{D}_{\Gamma,\alpha}((r : a)(A_0, \ldots, A_{n-1}) =$

$$\begin{array}{ll} \textbf{if} & \forall i \in I_r \quad \mathcal{D}_{\Gamma,\alpha}(A_i) = p_i \xrightarrow{b_i} p_i' \quad \text{and} \quad \phi_r(b, a) \\ & \forall i \notin I_r \quad A_i = p_i \in X \\ \textbf{then} & f_\alpha(p_0, \ldots, p_{n-1}) \xrightarrow{a} (C_r)_\alpha(q_0, \ldots, q_{n-1}) \\ & \text{where} \quad q_j = \begin{cases} p_j & \text{si} \quad j \notin I_r \\ p_j' & \text{si} \quad j \in I_r \end{cases} \\ \textbf{else} & \perp \end{array}$$

[2] We write $\mathcal{D}_{\Gamma,\alpha}(A) = \perp$ when $\mathcal{D}_{\Gamma,\alpha}$ is not defined for A, i.e. when A doesnot correspond to any proof.

For α a **T**-algebra on X and $f \in Tn$ an n-ary operator, f_α stands for the derived operator for f w.r.t. α. It is the mapping $f_\alpha : X^n \to X$ given by $f_\alpha(v) = (\alpha \circ Tv)(f)$. We shall write $\Gamma, \alpha \vdash A : p \xrightarrow{a} p'$ when $\mathcal{D}_{\Gamma,\alpha}(A) = p \xrightarrow{a} p'$ and $\Gamma, \alpha \vdash p \xrightarrow{a} p'$ when such a proof tree A exists.

Proposition 9 (fonctoriality)
If $<f, \varphi>: (X, \alpha, \Gamma) \longrightarrow (Y, \beta, \Delta)$ *is a morphism in* $\mathcal{K}$ *then*

$$\Gamma, \alpha \vdash A : p \xrightarrow{a} p' \quad \text{entails} \quad \Delta, \beta \vdash A <f, \varphi>: \varphi(p) \xrightarrow{a} \varphi(p')$$

where $A <f, \varphi>$ *is the image of* A *through the mapping*

$$\mathbf{AR\text{-}P}(f, \varphi) : \mathbf{AR\text{-}P}(\mathcal{N}_\Gamma, X) \longrightarrow \mathbf{AR\text{-}P}(\mathcal{N}_\Delta, Y)$$

that replace the occurrences of variables $n \in \mathcal{N}_\Gamma$ *by those of* $f(n) \in \mathcal{N}_\Delta$ *and the occurrences of the null-ary operators* $x \in X$ *by those of* $\varphi(x) \in Y$.

Proof :
The mapping $\mathbf{AR\text{-}P}(f, \varphi)$ is inductively defined by

$$\begin{array}{rcll} A <f,\varphi> & = & <f(n)> & \textbf{if} \;\; A = <n> \quad \text{where} \;\; n \in \mathcal{N}_\Gamma \\ & = & \varphi(x) & \textbf{if} \;\; A = x \in X \\ & = & (r:a)(A_0 <f,\varphi>, \ldots, A_{n-1} <f,\varphi>) & \textbf{if} \;\; A = (r:a)(A_0, \ldots, A_{n-1}) \end{array}$$

The proof is by induction on the strcuture of A :

1. **Base case**
   Let $A = < n >$ be an atomic proof tree (i.e. $n \in \mathcal{N}_\Gamma$). Since $< f, \varphi >: (X, \alpha, \Gamma) \longrightarrow (Y, \beta, \Delta)$ is a morphism in $\mathcal{K}$, one has

   $$\Gamma_n = p \xrightarrow{a} p' \quad \Longrightarrow \quad \Delta_{f(n)} = \varphi(p) \xrightarrow{a} \varphi(p')$$

   Besides, $A <f, \varphi> = <f(n)>$. It follows

   $$\Delta, \beta \vdash A <f, \varphi>: \varphi(p) \xrightarrow{a} \varphi(p')$$

2. **General case**
   The proof tree is of the form $A = (r : a)(A_0, \ldots, A_{n-1})$ and for every index $i \in I_r$, the proof tree A_i corresponds to a proof

   $$\Gamma, \alpha \vdash A_i : p_i \xrightarrow{b_i} p_i'$$

   with the n-tuple of action b verifying the condition $\phi_r(b, a)$. For the other index the proof A_i corresponds to an element in X : $A_i = x_i \in X$. The elements p and p' are then given by

$$p = f_\alpha(p_0, \ldots, p_{n-1}) \quad \text{and} \quad p' = (C_r)_\alpha(q_0, \ldots, q_{n-1}) \quad \text{where} \quad q_j = \begin{cases} p_j & \textbf{if} \quad j \notin I_r \\ p_j' & \textbf{if} \quad j \in I_r \end{cases}$$

By induction hypothesis we deduce

$$\Delta, \beta \vdash A_i < f, \varphi >: \varphi(p_i) \xrightarrow{b_i} \varphi(p_i') \quad \text{for} \quad i \in I_r$$

Moreover, for $i \notin I_r$ one has $A_i < f, \varphi >= \varphi(x_i) \in Y$ so it follows

$$\Delta, \beta \vdash A < f, \varphi >: f_\beta(\varphi(p_0), \ldots, \varphi(p_{n-1})) \xrightarrow{a} (C_r)_\beta(\varphi(q_0), \ldots, \varphi(q_{n-1}))$$

Since φ is a $\mathbf{T}$-algebra morphism from (X, α) to (Y, β) :

$$\begin{array}{rcccl} f_\beta(\varphi(p_0), \ldots, \varphi(p_{n-1})) & = & \varphi(f_\alpha(p_0, \ldots, p_{n-1})) & = & \varphi(p) \\ (C_r)_\beta(\varphi(q_0), \ldots, \varphi(q_{n-1})) & = & \varphi((C_r)_\alpha(q_0, \ldots, q_{n-1})) & = & \varphi(p') \end{array}$$

Hence,

$$\Delta, \beta \vdash A < f, \varphi >: \varphi(p) \xrightarrow{a} \varphi(p')$$

□

We so obtain an endofunctor $\mathcal{D}$ on the category $\mathcal{K}$ given by :

$$\mathcal{D} : \mathcal{K} \longrightarrow \mathcal{K} \; = \; \left\{ \begin{array}{lcl} (X, \alpha, \Gamma) & \mapsto & (X, \alpha, \mathcal{D}_{\Gamma,\alpha}) \\ <\varphi, f> & \mapsto & <\varphi, \mathbf{AR\text{-}P}(f, \varphi)> \end{array} \right.$$

Moreover the identities

1. $\mathcal{D}_{\Gamma,\alpha} \circ \mathrm{e}_{\mathcal{N}_\Gamma, X} = \Gamma$
2. $\mathcal{D}_{\Gamma,\alpha} \circ \mathrm{m}_{\mathcal{N}_\Gamma, X} = \mathcal{D}_{\mathcal{D}_{\Gamma,\alpha},\alpha}$

that are mainly a recasting of the definition of $\mathcal{D}_{\Gamma,\alpha}$ allows us to define the two morphisms

$$E_{X,\alpha,\Gamma} = < 1_X, \mathrm{e}_{\mathcal{N}_\Gamma, X} >: (X, \alpha, \Gamma) \longrightarrow \mathcal{D}(X, \alpha, \Gamma)$$

and

$$M_{X,\alpha,\Gamma} = < 1_X, \mathrm{m}_{\mathcal{N}_\Gamma, X} >: \mathcal{D}\mathcal{D}(X, \alpha, \Gamma) \longrightarrow \mathcal{D}(X, \alpha, \Gamma)$$

for which

Proposition 10 *$(\mathcal{D}, E, M)$ is a monad on the category $\mathcal{K}$.*

Proof : Follows immediatly from the fact that $(\mathbf{AR\text{-}P}(-, X), \mathrm{e}_{-,X}, \mathrm{m}_{-,X})$ is a monad.

□

An interesting consequence of the previous result is the following expansion rule that involved distinct $\mathbf{T}$-algebra structures. Modulo the two $\mathbf{T}$-algebras and the morphism (φ) that relates them, that rule says that if every assumption of Δ has a proof under assumptions in Γ, then any proof under assumptions in Δ can be expanded into a proof under assumptions in Γ.

Corollary 11 (expansion rule)
Let (X,α,Γ) and (Y,β,Δ) be two objects in $\mathcal{K}$ and $\varphi : (Y,\beta) \to (X,\alpha)$ a morphism of $\mathbf{T}$-algebras. We suppose that for any assumption $n \in \mathcal{N}_\Delta$ (with $\Delta(n) = q \xrightarrow{b} q'$) we have a proof tree $B_n \in AR\text{-}P(\mathcal{N}_\Gamma, X)$ for which

$$\Gamma, \alpha \vdash B_n : \varphi(q) \xrightarrow{b} \varphi(q')$$

then to any proof under assumptions in Δ

$$\Delta, \beta \vdash A : p \xrightarrow{a} p'$$

we can associate the proof under assumptions in Γ

$$\Gamma, \alpha \vdash A[B,\varphi] : \varphi(p) \xrightarrow{a} \varphi(p')$$

where the proof tree $A[B,\varphi]$ is obtain from A by replacing each occurrence of a variable $n \in \mathcal{N}_\Delta$ by the corresponding tree B_n and each occurrence of an null-ary operator $y \in Y$ by the operator $\varphi(y) \in X$.

Note : The reader may have noticed the similarity of that result with the proposition stating the functoriality of $\mathcal{D}$. And actually, it follows from that proposition thanks to the substitution operation $M_{X,\alpha,\Gamma}$. In more details
Proof : The hypothesis of the above result express the fact that

$$<B,\varphi>: (Y,\beta,\Delta) \longrightarrow (X,\alpha,\mathcal{D}_{\Gamma,\alpha}) = \mathcal{D}(X,\alpha,\Gamma)$$

is a morphism in the category $\mathcal{K}$. Let

$$<B,\varphi>^* = \mathcal{D}(Y,\beta,\Delta) \xrightarrow{\mathcal{D}(<B,\varphi>)} \mathcal{D}\mathcal{D}(X,\alpha,\Gamma) \xrightarrow{M_{X,\alpha,\Gamma}} \mathcal{D}(X,\alpha,\Gamma)$$

be its inductive extension. We recall that $<B,\varphi>^*$ is the unique morphism that extends $<B,\varphi>$ to $\mathcal{D}(Y,\beta,\Delta)$, that is to say such that

$$<B,\varphi> = <B,\varphi>^* \circ E_{Y,\beta,\Delta}$$

$$\begin{tikzcd}
(Y,\beta,\Delta) \arrow[r, "E_{Y,\beta,\Delta}"] \arrow[dr, "<B,\varphi>"'] & (Y,\beta,\mathcal{D}_{\Delta,\beta}) \arrow[d, "<B,\varphi>^*"] \arrow[dr, "\mathcal{D}(<B,\varphi>)"] & \\
& (X,\alpha,\mathcal{D}_{\Gamma,\alpha}) & \mathcal{D}\mathcal{D}(X,\alpha,\Gamma) \arrow[l, "M_{X,\alpha,\Gamma}"]
\end{tikzcd}$$

And $<B,\varphi>^* = M_{X,\alpha,\Gamma} \circ \mathcal{D}(<B,\varphi>) = <1_X, \mathfrak{m}_{\mathcal{N}_\Gamma,X}> \circ <1_X, \mathbf{AR\text{-}P}(B,\varphi)> =$
$= <1_X, \mathfrak{m}_{\mathcal{N}_\Gamma,X} \circ \mathbf{AR\text{-}P}(B,\varphi)>$. Hence the result with

$$A[B,\varphi] = (\mathfrak{m}_{\mathcal{N}_\Gamma,X} \circ \mathbf{AR\text{-}P}(B,\varphi))(A) = \mathfrak{m}_{\mathcal{N}_\Gamma,X}(A <B,\varphi>)$$

Remark : As a comparison with $A < B, \varphi >$, the mapping

$$(.)[B,\varphi] : \mathbf{AR\text{-}P}(\mathcal{N}_\Delta, Y) \longrightarrow \mathbf{AR\text{-}P}(\mathcal{N}_\Gamma, X)$$

is inductively defined by :

$$\begin{array}{rcll} A[B,\varphi] & = & B_n & \textbf{if } A = <n> \quad \text{with} \quad n \in \mathcal{N}_\Gamma \\ & = & \varphi(x) & \textbf{if } A = x \in X \\ & = & (r:a)(A_0[B,\varphi], \ldots, A_{n-1}[B,\varphi]) & \textbf{if } A = (r:a)(A_0, \ldots, A_{n-1}) \end{array}$$

□

3.3 The operational model

In this paragraph we show how the operational models can be viewed as the algebras for a monad on a category of transition systems. We have already define a transition system on a set X as a subset $\xi \subseteq X \times A \times X$ and we are familiar with the notation $x \xrightarrow{a} x' \in \xi$ for $< x, a, x' > \in \xi$; now in order to get a category we let a morphism of transition systems $f : (X, \xi) \to (Y, \zeta)$ be any mapping $f : X \to Y$ between the underlying sets for which

$$x \xrightarrow{a} x' \in \xi \quad \Longrightarrow \quad f(x) \xrightarrow{a} f(x') \in \zeta$$

The category of transition systems so obtained is a particular case of category of sets with structure.

Definition 12 *A (small) category of sets with structure $\mathcal{C}$ consists in the following data. Firstly, for each set X we have a set $\mathcal{C}(X)$ of* $\mathcal{C}$**-structures on** *X and for each pair $(\xi, \zeta) \in \mathcal{C}(X) \times \mathcal{C}(Y)$ of $\mathcal{C}$-structures we have a subset of mappings from X to Y said to be* $\mathcal{C}$**-admissible** *from (X, ξ) to (Y, ζ). The two following conditions must be fulfilled*

1. *stability by composition :*
   if $f : (X, \xi) \to (Y, \zeta)$ and $g : (Y, \zeta) \to (Z, \kappa)$ are both admissible then $g \circ f : (X, \xi) \to (Z, \kappa)$ must as well be admissible.

2. *transport of structures :*
   If $f : X \to Y$ is a bijection and $\xi \in \mathcal{C}(X)$ a $\mathcal{C}$-structure on X, then there exists a unique $\mathcal{C}$-structure $\zeta \in \mathcal{C}(Y)$ on Y for which both $f : (X, \xi) \to (Y, \zeta)$ and $f^{-1} : (Y, \zeta) \to (X, \xi)$ are admissible mappings.

As a consequence of the previous axioms we note that the identity mapping $1_X : (X, \xi) \to (X, \xi)$ is admissible for every $\mathcal{C}$-structure ξ on X. In this way $\mathcal{C}$ can be construe as a category whose objects are the pairs (X, ξ) with $\xi \in \mathcal{C}(X)$ and whose arrows are the mapping $\mathcal{C}$-admissible. Moreover we have forgetful functor that erases the $\mathcal{C}$-structure

$$U(X, \xi) = X \quad \text{and} \quad U(f) = f.$$

In the above definition we can replace the category of set by any base category $\mathcal{B}$ leading to the more general notion of category of $\mathcal{B}$-objects with structure.

Here we are only concerned by algebraic structures that are extensions of some algebraic structures defined on sets. More precisely,

Definition 13 *Let* $\mathbf{T}$ *be a (finitary) monad on the category of sets, a* **lift** *of* $\mathbf{T}$ *to a category* $U : \mathcal{C} \to \mathbf{Set}$ *of sets with structure is a monad* $\tilde{\mathbf{T}} = (\tilde{T}, \tilde{\eta}, \tilde{\mu})$ *on the category* $\mathcal{C}$ *such that*

$$U\tilde{T} = TU \quad ; \quad U\tilde{\eta} = \eta_U \quad \textit{and} \quad U\tilde{\mu} = \mu_U$$

Such a lift amounts to a $\mathcal{C}$-structure $\tilde{T}(\xi)$ on TX for every $\mathcal{C}$-structure $\xi \in \mathcal{C}(X)$ with the following requirements

1. *fonctoriality*
   For every $\mathcal{C}$-admissible mapping f from (X, ξ) to (Y, ζ), Tf is $\mathcal{C}$-admissible from $(TX, \tilde{T}\xi)$ to $(TY, \tilde{T}\zeta)$.

2. *compatibility with the atomic structures*
   For every $\mathcal{C}$-structure $\xi \in \mathcal{C}(X)$, η_X is $\mathcal{C}$-admissible from (X, ξ) to $(TX, \tilde{T}\xi)$.

3. *compatibility with the substitution operation*
   For every $\mathcal{C}$-structure $\xi \in \mathcal{C}(X)$, μ_X is $\mathcal{C}$-admissible from $(TTX, \tilde{T}\tilde{T}\xi)$ to $(TX, \tilde{T}\xi)$.

The corresponding lift $\tilde{\mathbf{T}} = (\tilde{T}, \tilde{\eta}, \tilde{\mu})$ for $\mathbf{T}$ is then given by

$$\begin{array}{ll}
\tilde{T}(X,\xi) = (TX, \tilde{T}\xi) & \text{for} \quad \xi \in \mathcal{C}(X) \\
\tilde{T}(f) = T(f) & \text{for} \quad f \in \mathcal{C}((X,\xi);(Y,\zeta)) \\
\tilde{\eta}_{(X,\xi)} = \eta_X & \text{for} \quad \xi \in \mathcal{C}(X) \\
\tilde{\mu}_{(X,\xi)} = \mu_X & \text{for} \quad \xi \in \mathcal{C}(X)
\end{array}$$

Moreover, the category of algebras $\mathcal{C}^{\tilde{\mathbf{T}}}$ is as well isomorphic to a category of sets with structure whose objects (X, α, ξ) are sets X equipped with a $\mathbf{T}$-algebra structure α and a $\mathcal{C}$-structure $\xi \in \mathcal{C}(X)$; those two structure being compatible in the sense that α is a $\mathcal{C}$-admissible mapping from $\tilde{T}(X, \xi)$ to (X, ξ). A mapping $\varphi : X \to Y$ is admissible from (X, α, ξ) to (Y, β, ζ) whenever it is a $\mathbf{T}$-algebra morphism $\varphi : (X, \alpha) \to (Y, \beta)$, $\mathcal{C}$-admissible from (X, ξ) to (Y, ζ).

For instance we obtain a lift for the algebraic structure of terms to a category of transition systems by letting $\tilde{T}\xi \subseteq TX \times A \times TX$ be the set of transitions provable in the deductive system obtained by adding to **SOS** the transitions of ξ as additional axioms.

$$t \xrightarrow{a} t' \in \tilde{T}\xi \quad \Longleftrightarrow \quad \mathbf{SOS}, \xi \vdash t \xrightarrow{a} t'$$

More precisely, we add the axiom $< x > \xrightarrow{a} < x' >$ for each transition $x \xrightarrow{a} x' \in \xi$. That means that the elements of $\tilde{T}\xi$ are the transitions $t \xrightarrow{a} t' \in TX \times A \times TX$ such that

$$\ulcorner \xi \urcorner, \mu_X \vdash t \xrightarrow{a} t'$$

i.e. provable in the free algebra (TX, μ_X) under the assumptions

$$\ulcorner \xi \urcorner : \xi \longrightarrow TX \times A \times TX$$

where

$$\ulcorner \xi \urcorner (x \xrightarrow{a} x') = < x > \xrightarrow{a} < x' >$$

For example, if $x \xrightarrow{a} x'$ and $z \xrightarrow{\bar{a}} z'$ are in ξ, both following transitions are in $\tilde{T}\xi$

$$< x > \xrightarrow{a} < x' > \qquad \text{and} \qquad \begin{tikzcd}[column sep=small, row sep=small] & & \| \arrow[dl, dash] \arrow[dr, dash] & & & & \| \arrow[dl, dash] \arrow[dr, dash] & \\ & + \arrow[dl, dash] \arrow[dr, dash] & & <z> \arrow[rr, "\tau"] & & <x'> & & <z'> \\ <x> & & <y> & & & & & \end{tikzcd}$$

We first check that we actually obtain a lift of $\mathbf{T}$:

Proposition 14 *We obtain a lift $\tilde{\mathbf{T}}$ of the algebraic structure of terms to the category $\mathcal{C}$ of transition systems, by letting $\tilde{T}\xi$ be the set of provable transitions in the free algebra (TX, μ_X) under the assumptions $\ulcorner \xi \urcorner$.*

Proof : We check the three requirements :

1. *For every admissible mapping $f : (X, \xi) \to (Y, \zeta)$, Tf is admissible from $(TX, \tilde{T}\xi)$ to $(TY, \tilde{T}\zeta)$.*
   For this, consider a transition $t \xrightarrow{a} t' \in \tilde{T}\xi$, i.e.

   $$\ulcorner \xi \urcorner, \mu_X \vdash A : t \xrightarrow{a} t'$$

   due to the naturality of η we know that the following diagram commutes

   $$\begin{tikzcd} \xi \arrow[r, "\ulcorner \xi \urcorner"] \arrow[d, "f^\circ"'] & TX \times A \times TX \arrow[d, "Tf \times 1_A \times Tf"] \\ \zeta \arrow[r, "\ulcorner \zeta \urcorner"] & TY \times A \times TY \end{tikzcd} \qquad \begin{tikzcd} (TX, \mu_X) \arrow[d, "Tf"] \\ (TY, \mu_Y) \end{tikzcd}$$

   where f° stands for the mapping $f \times A \times f$ restricted to $\xi \subset X \times A \times X$ and co-restricted to $\zeta \subset Y \times A \times Y$. Since moreover Tf is a $\mathbf{T}$-algebra morphism from (TX, μ_X) to (TY, μ_Y) we get that

   $$< f^\circ, Tf >: (TX, \mu_X, \ulcorner \xi \urcorner) \longrightarrow (TY, \mu_Y, \ulcorner \zeta \urcorner)$$

is a morphism of the category $\mathcal{K}$. Thanks to the functoriality of $\mathcal{D}$ we deduce

$$\ulcorner\zeta\urcorner, \mu_Y \vdash A <f, Tf>: Tf(t) \xrightarrow{a} Tf(t')$$

hence $Tf(t) \xrightarrow{a} Tf(t') \in \tilde{T}\zeta$.

2. *η_X is admissible from (X, ξ) to $(TX, \tilde{T}\xi)$.*
   This comes from the fact that for every transition $x \xrightarrow{a} x' \in \xi$ one has

$$\ulcorner\xi\urcorner, \mu_X \vdash <x \xrightarrow{a} x'>:<x>\xrightarrow{a}<x'>$$

3. *μ_X is admissible from $(TTX, \tilde{T}\tilde{T}\xi)$ to $(TX, \tilde{T}\xi)$.*
   For each transition $\mathcal{T} = t \xrightarrow{a} t'$ in $\tilde{T}\xi$ we choose a proof tree $B_{\mathcal{T}} \in$ AR-P(ξ, TX) for which

$$\ulcorner\xi\urcorner, \mu_X \vdash B_{\mathcal{T}} : t \xrightarrow{a} t'$$

   Since the composite $\mu \circ \eta_{T}$ is the identity and since μ_X is a $\mathbf{T}$-algebra morphism from (TTX, μ_{TX}) to (TX, μ_X), we get that

$$<B, \mu_X>: (TTX, \mu_{TX}, \ulcorner\tilde{T}\xi\urcorner) \longrightarrow (TX, \mu_X, \mathcal{D}_{\ulcorner\xi\urcorner, \mu_X})$$

   is a morphism of the category $\mathcal{K}$. Now let a transition $u \xrightarrow{a} u' \in \tilde{T}\tilde{T}\xi$ be given, we have therefore a proof

$$\ulcorner\tilde{T}\xi\urcorner, \mu_{TX} \vdash \mathcal{A} : u \xrightarrow{a} u' \in TTX \times A \times TTX$$

   from which we can, thanks to the expansion rule, deduce the proof

$$\ulcorner\xi\urcorner, \mu_X \vdash \mathcal{A}[B, \mu_X] : \mu_X(u) \xrightarrow{a} \mu_X(u') \in TX \times A \times TX$$

   Hence $\mu_X(u) \xrightarrow{a} \mu_X(u') \in \tilde{T}\xi$ as required.

□

We recall that the adequacy of a premodel w.r.t. the operational specifications is given by

$$\mathbf{SOS}, \xi \vdash t \xrightarrow{a} t' \quad \Longrightarrow \quad (X, \alpha, \xi) \models t \xrightarrow{a} t'$$

which can be restated as

$$t \xrightarrow{a} t' \in \tilde{T}\xi \quad \Longrightarrow \quad \alpha(t) \xrightarrow{a} \alpha(t') \in \xi$$

which in turn expresses the admissibility of α from $(TX, \tilde{T}\xi)$ to (X, ξ), property that characterizes the algebras of $\tilde{\mathbf{T}}$. Hence,

Proposition 15 (full adequacy)
The algebras for the monad $\tilde{\mathbf{T}}$ are the operational models.

We can use the above characterization of the operational models as algebras, to get an equivalent formulation of the adequacy of a premodel that may be more in accordance with the usual definition. For that purpose, if $t \in TY$ is a term and $v : Y \to X$ une valuation we let

$$[\![t]\!]_{\alpha,v} = (\alpha \circ Tv)(t)$$

be the *value* of t in the *valued interpretation* $(X, \alpha; v)$. And we define a relation of *validation*

$$\mathcal{M}, v \models t \xrightarrow{a} t' \quad \Longleftrightarrow \quad [\![t]\!]_{\alpha,v} \xrightarrow{a} [\![t']\!]_{\alpha,v} \in \xi$$

that generalizes the relation we already introduced (which corresponds to the case where v is the identity mapping) :

$$(X, \alpha, \xi) \models t \xrightarrow{a} t' \in TX \times A \times TX \quad \Longleftrightarrow \quad \alpha(t) \xrightarrow{a} \alpha(t') \in \xi$$

We then have the alternative characterization of the operational models :

Proposition 16 *A premodel $\mathcal{M} = (X, \alpha, \xi)$ is an operational model if, and only if, for every set Y and transition system $\zeta \subset Y \times A \times Y$ the fact that a transition $t \xrightarrow{a} t' \in TY \times A \times TY$ be provable with* **SOS** *under the assumptions ζ :*

$$\mathbf{SOS}, \zeta \vdash t \xrightarrow{a} t'$$

entails its validity in $\mathcal{M}$

$$\mathcal{M}, v \models t \xrightarrow{a} t'$$

for every valuation $v \in X^Y$ satisfying ζ, i.e. such that v is admissible from (Y, ζ) in (X, ξ).

Proof : Let $\mathcal{M} = (X, \alpha, \xi)$ be a premodel verifying the above condition, we in particular (when $Y = X$, $\zeta = \xi$ and $v = 1_X$) get

$$\mathbf{SOS}, \xi \vdash t \xrightarrow{a} t' \quad \Longrightarrow \quad (X, \alpha, \xi) \models t \xrightarrow{a} t'$$

i.e.

$$t \xrightarrow{a} t' \in \tilde{T}\xi \quad \Longrightarrow \quad \alpha(t) \xrightarrow{a} \alpha(t') \in \xi$$

which means that $\mathcal{M} = (X, \alpha, \xi)$ is a $\tilde{\mathbf{T}}$-algebra. Conversely, let (X, α, ξ) be a $\tilde{\mathbf{T}}$-algebra, $t \xrightarrow{a} t' \in TY \times A \times TY$ a transition provable under the assumption ζ

$$\mathbf{SOS}, \zeta \vdash t \xrightarrow{a} t' \quad \text{i.e.} \quad t \xrightarrow{a} t' \in \tilde{T}\zeta$$

and $v : (Y, \zeta) \to (X, \xi)$ an admissible mapping. Thanks to the adjunction situation corresponding to the Eilenberg-Moore construction for $\tilde{\mathbf{T}}$ we know that

$$\psi = \alpha \circ Tv : (TY, \mu_Y, \tilde{T}\zeta) \longrightarrow (X, \alpha, \xi)$$

is the unique morphism in $\mathcal{C}^{\tilde{\mathbf{T}}}$ giving rise to the following factorisation in $\mathcal{C}$

$$v = (Y, \zeta) \xrightarrow{\eta_Y^{(\mathbf{T})}} (TY, \tilde{T}\zeta) \xrightarrow{\psi} (X, \xi)$$

From $t \xrightarrow{a} t' \in \tilde{T}\zeta$ we deduce $\psi(t) \xrightarrow{a} \psi(t') \in \xi$ thanks to the admissibility of ψ. And consequently, $\mathcal{M}, v \models t \xrightarrow{a} t'$ (for $\psi(t)$ is none other than $[\![t]\!]_{\alpha,v}$).
□
The adjunction that appears in the previous proof describes the free algebra $(TX, \mu_X, \tilde{T}\xi)$ as the '*least operational model verifying the assumptions* ξ' in the same way as the algebra (TX, μ_X) is the free **T**-algebra containing the set of variables X.

4 The category of lifts

The above characterization of the operational models as algebras, put emphasize on algebraic structures (monads) that are lift of algebraic structures on sets to categories of sets with structure. Provided those additional structures satisfy some completness properties (to be defined in the paragraph (4.1)) we will show that the algebraic structure and the additional structure can be used independently to define quotients.

4.1 Fibre complete categories of sets with structure

In this section we define the completness properties that will be assumed for the categories of sets with structure. If $\mathcal{C}$ is any category of $\mathcal{B}$-objects with structure, we can define an order relation on the set $\mathcal{C}(X)$ of $\mathcal{C}$-structure of a $\mathcal{B}$-object X by letting $\xi \leq \zeta$ when the identity 1_X is admissible from (X, ξ) to (X, ζ) (the axiom of transport of structure ensures the antisymmetry). As for the category of transition systems this relation is given by the set inclusion and therefore each *fibre* $(\mathcal{C}(X), \leq)$ is a complete ordered set. The category of transition systems satisfies as well another completness property : let $f : X \to Y$ be any mapping, then for any transition system $\xi \in \mathcal{C}(X)$ on X there exists a *minimal* transition system $\zeta \in \mathcal{C}(Y)$ for which $f : (X, \xi) \to (Y, \zeta)$ is admissible, given by

$$f^*(\xi) = \{f(x) \xrightarrow{a} f(x') \mid x \xrightarrow{a} x' \in \xi\}$$

Conversely for any structure $\zeta \in \mathcal{C}(Y)$ there exists a *maximal* structure $\xi \in \mathcal{C}(X)$ on X making $f : (X, \xi) \to (Y, \zeta)$ admissible, given by

$$f_*(\xi) = \{x \xrightarrow{a} x' \mid f(x) \xrightarrow{a} f(x') \in \xi\}$$

Definition 17 *Let $\mathcal{C}$ be a category of $\mathcal{B}$-objects with structure, $f : X \to Y$ an arrow of the base category and $\xi \in \mathcal{C}(X)$ a $\mathcal{C}$-structure on X. The* **inverse image** *of*

ξ along f, when it exists, is the least $\mathcal{C}$-structure $f^*(\xi) \in \mathcal{C}(Y)$ on Y making f admissible; that means this structure is characterized by the equivalence

$$f : (X, \xi) \to (Y, \zeta) \text{ admissible} \quad \Longleftrightarrow \quad f^*(\xi) \leq \zeta$$

In the same manner, the **direct image** *$f_*(\zeta)$ of a $\mathcal{C}$-structure $\zeta \in \mathcal{C}(Y)$ is given by*

$$f : (X, \xi) \to (Y, \zeta) \text{ admissible} \quad \Longleftrightarrow \quad \xi \leq f_*(\zeta)$$

If f has both an inverse image f^* and a direct image f_*, we have for every pair of $\mathcal{C}$-structures $(\xi, \zeta) \in \mathcal{C}(X) \times \mathcal{C}(Y)$ the equivalence

$$f^*(\xi) \leq \zeta \quad \Leftrightarrow \quad \xi \leq f_*(\zeta)$$

which amounts to the following adjunction situation

$$f^* \dashv f_* : \mathcal{C}(Y) \to \mathcal{C}(X)$$

in which the ordered sets $(\mathcal{C}(X), \leq)$ are viewed as categories. It is easy to check that an inverse (or direct) image when it exists is unique and moreover, as in any adjunction situation, any of the inverse (or direct) image determines the other **assuming both exist**. More precisely, the formal criterion for the existence of an adjoint [Lan71, page 230] gives in the particular case of ordered sets the following result

Proposition 18 *We suppose that every fibre $(\mathcal{C}(X), \leq)$ is complete and we consider an arrow $f : X \to Y$ in the base category. If f has an inverse image $f^* : \mathcal{C}(X) \to \mathcal{C}(Y)$ that preserves least upper bounds i.e. such that*

$$f^*(\bigvee \xi_i) = \bigvee f^*(\xi_i)$$

then f has as well a directe image $f_* : \mathcal{C}(Y) \to \mathcal{C}(X)$ given by

$$f_*(\zeta) = \bigvee \{\xi \in \mathcal{C}(X) \;\; ; \;\; f^*(\xi) \leq \zeta\}$$

Dually, if a direct image preserving greatest lower bounds exists for f, then f has an inverse image defined by

$$f^*(\xi) = \bigwedge \{\zeta \in \mathcal{C}(Y) \;\; ; \;\; \xi \leq f_*(\zeta)\}$$

Now we define the categories of sets with structure we shall be interested in.

Definition 19 *A category $U : \mathcal{C} \to \mathcal{B}$ of $\mathcal{B}$-objects with structure is said to be* **fibre-complete** *when*

1. *each of its fibre $(\mathcal{C}(X), \leq)$ is a complete ordered set, i.e. has arbitrary least upper bounds and greatest lower bounds; and*

2. *every arrow $f : X \to Y$ of $\mathcal{B}$ has an inverse image $f^* : \mathcal{C}(X) \to \mathcal{C}(Y)$ and a direct image $f_* : \mathcal{C}(Y) \to \mathcal{C}(X)$.*

4.2 The category of lifts

We shall be concerned only by lifts for algebraic structures over sets to fibre-complete categories of sets with structure.

Definition 20 *The category $\mathcal{L}ifts$ is a category of $\mathcal{B}$-objects with structure. The base category $\mathcal{B} = \mathbf{Fin}\text{-}\mathcal{M}on(\mathbf{Set}) \times \mathbf{FC}\text{-}\mathcal{S}truc(\mathbf{Set})^{op}$ is the cartesian product of the category of finitary monads over set with the dual of the category of fibre-complete categories of sets with structure. The set $\mathcal{L}ifts(\mathbf{T}, \mathcal{C})$ is made of the lifts $\tilde{\mathbf{T}}$ for the monad $\mathbf{T}$ to the category $\mathcal{C}$. An arrow*

$$<\varphi, V>:<\mathbf{T}, \mathcal{C}> \longrightarrow <\mathbf{S}, \mathcal{D}>$$

in the base category $\mathcal{B}$ is made of a monad morphism $\varphi : \mathbf{T} \to \mathbf{S}$ together with a functor $V : \mathcal{D} \to \mathcal{C}$ that commutes with the forgetful functors (i.e. it respects the underlying sets)

$$\begin{tikzcd}
\mathbf{T} \arrow[d, "\varphi"] & \mathcal{C} \arrow[dr, "U"] & \\
 & & \mathbf{Set} \\
\mathbf{S} & \mathcal{D} \arrow[uu, "V"] \arrow[ur, "U'"'] &
\end{tikzcd}$$

Such an arrow is said to be admissible from $< \mathbf{T}, \mathcal{C}, \tilde{\mathbf{T}} >$ to $< \mathbf{S}, \mathcal{D}, \tilde{\mathbf{S}} >$ (where $\tilde{\mathbf{T}} \in \mathcal{L}ifts(\mathbf{T}, \mathcal{C})$ and $\tilde{\mathbf{S}} \in \mathcal{L}ifts(\mathbf{S}, \mathcal{D})$) when the following condition is fulfilled

> *for all $\mathcal{D}$-structure $\xi \in \mathcal{D}(X)$, the mapping φ_X is $\mathcal{C}$-admissible from $(TX, \tilde{T}V\xi)$ to $(SX, V\tilde{S}\xi)$.*

(for $\xi \in \mathcal{D}(X)$, $V(X, \xi)$ is of the form $(X, V\xi)$ since V respects the underlying sets).

The condition of admissibility in $\mathcal{L}ifts$ means that we obtain a generalized monad morphism $(V, \tilde{\varphi}) : (\mathcal{D}, \tilde{\mathbf{S}}) \to (\mathcal{C}, \tilde{\mathbf{T}}$ in which the components of $\tilde{\varphi} : \tilde{\mathbf{T}}V \to V\tilde{\mathbf{S}}$ are given by the φ-components : $\tilde{\varphi}_{(X,\xi)} = \varphi_X$.
The order relation on a fibre $\mathcal{L}ifts(\mathbf{T}, \mathcal{C})$ is given by

$$\tilde{\mathbf{T}} \leq \hat{\mathbf{T}} \iff \forall X \text{ et } \xi \in \mathcal{C}(X) \quad \tilde{T}\xi \leq \hat{T}\xi$$

The completness properties we assume for the categories on which our lifts are to be taken give completness results for the category of lifts itself. The following proposition, which proof is given in the appendix, state that any such lift may be transformed in a *minimal* way in another lift along a monad morphism and along some functors between the categories in which the algebras takes their values.

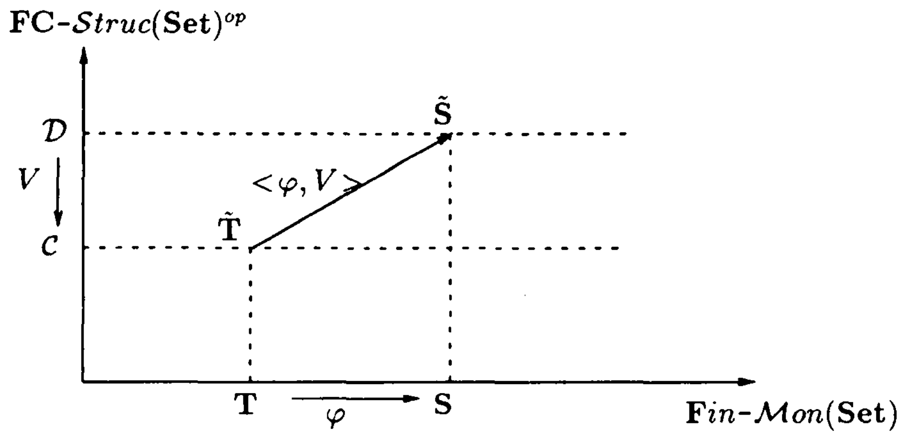

Figure 1: The category $\mathcal{L}ifts$

Proposition 21 *Each of the fibres of $\mathcal{L}ifts$ is complete. Moreover, if $V : \mathcal{D} \to \mathcal{C}$ is a functor that commutes with the forgetful functors and preserves the greatest lower bounds (i.e. for every family $\{\xi_i \ ; \ i \in I\}$ of $\mathcal{D}$-structures $\xi_i \in \mathcal{D}(X)$ on X one has $V(\bigwedge_{i\in I} \xi_i) = \bigwedge_{i\in I}(V\xi_i)$, then for all monad morphism $\varphi : \mathbf{T} \to \mathbf{S}$, the morphism $<\varphi, V>:<\mathbf{T}, \mathcal{C}> \longrightarrow <\mathbf{S}, \mathcal{D}>$ has an inverse image*

$$<\varphi, V>^*: \mathcal{L}ifts(\mathbf{T}, \mathcal{C}) \longrightarrow \mathcal{L}ifts(\mathbf{S}, \mathcal{D})$$

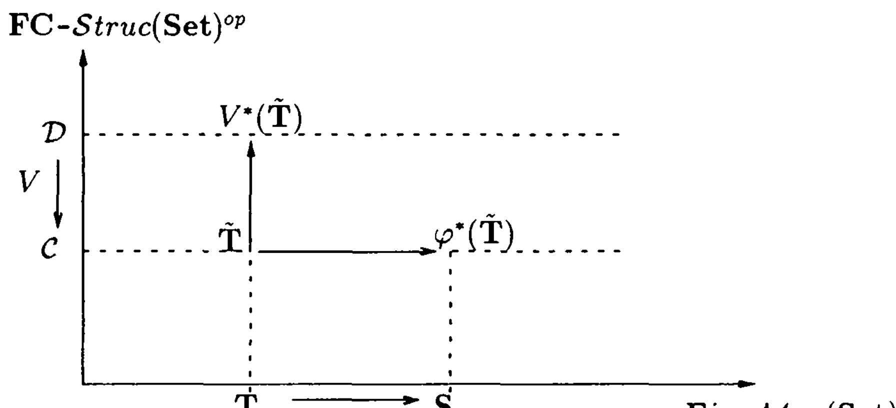

Figure 2: transports of lifts

In the same way, if V preserves the least upper bounds then $<\varphi, V>$ has a direct image.
We'll write $\varphi^* =<\varphi, 1_{\mathcal{C}}>^*$ and $V^* =< 1_{\mathbf{T}}, V>^*$ in short when one of the component is an identity. So, for any object $(\mathbf{T}, \mathcal{C}, \tilde{\mathbf{T}})$ of $\mathcal{L}ifts$ and monad morphism $\varphi : \mathbf{T} \to \mathbf{S}$, the lift $\tilde{\mathbf{T}}$ for $\mathbf{T}$ can be *transported* in a *minimal* manner along φ to give a lift $\tilde{\mathbf{S}} = \varphi^*(\tilde{\mathbf{T}})$ for $\mathbf{S}$. In the same way, every functor $V : \mathcal{D} \to \mathcal{C}$ that respects the

underlying sets and the greatest lower bounds on each fibre provides a *minimal* lift $\hat{\mathbf{T}} = V^*(\tilde{\mathbf{T}})$ for $\mathbf{T}$ to $\mathcal{D}$. As a consequence the category $\mathcal{L}ifts$ has a two-dimensional structure in the cartesian meaning of the word, since we have the ability to *move parallel to* both of the axis. In this way quotients may be defined in that category by adding constraints either on the underlying algebraic structure (e.g. by considering equational variety) or on the category in which the algebras takes their values.

4.2.1 Change of the underlying algebraic structure

Each morphism between two finitary algebraic structures on the category of sets has a uniform extension to the corresponding families of lifts, this enables one to work at the level of sets. For instance the lift $\tilde{\mathbf{T}}$ for the monad of terms to a category of transition systems that was derived from the operational specifications may be extended and provide a lift $\tilde{\mathbf{R}}$ for the theory of rational trees. $\tilde{\mathbf{R}}$ is the least element of the fibre $\mathcal{L}ifts(\mathbf{R}, \mathcal{C})$ for which each of the φ components are admissible mappings :

$$\varphi_X : (TX, \tilde{T}\xi) \longrightarrow (RX, \tilde{R}\xi)$$

(where $\varphi : \mathbf{T} \rightarrow \mathbf{R}$ is the embedding of the terms within the rational trees). Nevertheless this *minimal* extension of $\tilde{\mathbf{T}}$ to $\mathbf{R}$ (along φ) has a more direct characterization : $\tilde{R}\xi$ is made of the transitions that are provable in the deductive system obtained by adding to the operational rules (applied to the rational trees) the transitions of ξ as additional axioms. More precisely, let

$$\nu_X = TRX \xrightarrow{\varphi_{RX}} RRX \xrightarrow{\mu_X} RX$$

be the free $\mathbf{T}$-algebra structure on the rational trees (the Herbrand interpretation). By abuse of notation we shall also write $\ulcorner\xi\urcorner$ for the mapping

$$\ulcorner\xi\urcorner : \xi \longrightarrow RX \times A \times RX \quad \text{given by} \quad \ulcorner\xi\urcorner(x \xrightarrow{a} x') = <x> \xrightarrow{a} <x'>$$

(where this time $<x> = \eta_X^{(\mathbf{R})}(x)$ stands for the atomic rational tree associated to x) $\tilde{R}\xi$ is then characterized by

$$t \xrightarrow{a} t' \in \tilde{R}\xi \quad \Longleftrightarrow \quad \ulcorner\xi\urcorner, \nu_X \vdash t \xrightarrow{a} t'$$

Proposition 22 *If we let $\tilde{R}\xi$ be the set of transitions provable in the algebra (RX, ν_X) under the assumptions $\ulcorner\xi\urcorner$, we get a lift for $\mathbf{R}$ to the category $\mathcal{C}$ of transition systems. Moreover this is the least such lift for which each of the mappings φ_X be admissible from $(TX, \tilde{T}\xi)$ to $(RX, \tilde{R}\xi)$ (for all $\xi \in \mathcal{C}(X)$). i.e. $\tilde{\mathbf{R}}$ is the inverse image $\varphi^*(\tilde{\mathbf{T}})$ of $\tilde{\mathbf{T}}$ along φ.*

Proof : see the appendix.

Another example of transport of lifts along monad morphisms is obtained by looking at equational varieties of algebras. Let $\mathbf{T} = (T, \eta, \mu)$ be a finitary monad over the category of set, if (D, δ) is a $\mathbf{T}$-algebra and $v : V \to D$ a *valuation*, we can take each *syntactical expression* $e \in TV$ to the element in TD

$$e[v] = Tv(e)$$

got by replacing in e each variable in V by its v-value. Thanks to the $\mathbf{T}$-algebra structure we then obtain the *value* of this expression in the *valued interpretation* $(D, \delta; v)$. This value is therefore given by

$$v_\delta^*(e) = (\delta \circ Tv)(e)$$

A $\mathbf{T}$-equation is a pair $< e_1, e_2 >$ of elements in TV. A $\mathbf{T}$-algebra is said to satisfied such an equation when we have $v_\delta^*(e_1) = v_\delta^*(e_2)$ for any valuation $v : V \to D$. Let E be a set of $\mathbf{T}$-equation, we define the corresponding equational variety (**T**,E)-**Alg** as the full subcategory of $\mathbf{Set}^{\mathbf{T}}$ corresponding to the $\mathbf{T}$-algebras that satisfy each of the equations in E. We now (see e.g. [Man76]) that there exists a monad $\mathbf{T}_E$ on **Set** such that (**T**,E)-**Alg** and $\mathbf{Set}^{\mathbf{T}_E}$ are isomorphic, we let $\pi_E : \mathbf{T} \to \mathbf{T}_E$ be the corresponding monad morphism. Now, if $\tilde{\mathbf{T}}$ is a lift for $\mathbf{T}$, a $\tilde{\mathbf{T}}$-algebra (D, δ, ξ) is said to satisfied an equation $e_1 = = e_2$ (for $e_1, e_2 \in TV$) when its underlying $\mathbf{T}$-algebra (D, δ) satisfies it. Otherwise stated the subcategory $(\tilde{\mathbf{T}}, E)$-**Alg** of $\tilde{\mathbf{T}}$-algebra verifying E is given by the following pull-back

$$\begin{tikzcd}
(\tilde{\mathbf{T}}, E)\text{-}\mathbf{Alg} \arrow[r] \arrow[d] & \mathbf{Set}^{\mathbf{T}_E} \arrow[d, "\mathbf{Set}^{\pi_E}"] \\
\mathcal{C}^{\tilde{\mathbf{T}}} \arrow[r, "U^*"'] & \mathbf{Set}^{\mathbf{T}}
\end{tikzcd}$$

Now it appears that

Proposition 23 *The equational variety* $(\tilde{\mathbf{T}}, E)$-**Alg** *is algebraic over the category* $\mathcal{C}$*, which means that the forgetful functor*

$$U_E = (\tilde{\mathbf{T}}, E)\text{-}\mathbf{Alg} \longrightarrow \mathcal{C}^{\tilde{\mathbf{T}}} \xrightarrow{U^{\tilde{\mathbf{T}}}} \mathcal{C}$$

is monadic. Moreover the corresponding monad $\tilde{\mathbf{T}}_E$ *on* $\mathcal{C}$ *in none other than the inverse image of* $\tilde{\mathbf{T}}$ *along* π_E *:*

$$\tilde{\mathbf{T}}_E = \pi_E^*(\tilde{\mathbf{T}})$$

This proposition is in fact a particular case of the following result that gives a characterization of the algebras for the inverse image of a lift along a monad morphism in terms of that monad morphism and of the algebras for the initial lift.

Proposition 24 *Let* $\mathbf{T}$ *and* $\mathbf{S}$ *be two finitary monads on sets, for every lift* $\tilde{\mathbf{T}} \in \mathcal{L}ifts(\mathbf{T}, \mathcal{C})$ *for* $\mathbf{T}$ *and monad morphism* $\varphi : \mathbf{T} \to \mathbf{S}$ *we have a pull back*

$$\begin{tikzcd}
\mathcal{C}^{\tilde{\mathbf{S}}} \arrow[r, "U'^{*}"] \arrow[d, "\mathcal{C}^{\tilde{\varphi}}"'] & \mathbf{Set}^{\mathbf{S}} \arrow[d, "\mathbf{Set}^{\varphi}"] \\
\mathcal{C}^{\tilde{\mathbf{T}}} \arrow[r, "U^{*}"'] & \mathbf{Set}^{\mathbf{T}}
\end{tikzcd}$$

where

1. $\tilde{\mathbf{S}} = \varphi^*(\mathbf{S})$ *is the inverse image of* $\tilde{\mathbf{T}}$ *along* φ, *that is the least lift for* $\mathbf{S}$ *to* $\mathcal{C}$ *for which each of the component of* φ *is admissible,*

2. $\tilde{\varphi} : \tilde{\mathbf{T}} \longrightarrow \tilde{\mathbf{S}}$ *is the resulting monad morphism :*

$$\tilde{\varphi}_{(A,\xi)} = \varphi_A : (TA, \tilde{T}\xi) \longrightarrow (SA, \tilde{S}\xi) \quad \textit{and,}$$

3. U^* *et* U'^* *are the respective lifts for* $U : \mathcal{C} \to \mathbf{Set}$ *to the categories of algebras.*

Proof : see the appendix.

4.2.2 Change of the base category

In this paragraph we outline how quotients in the category $\mathcal{L}ifts$ may be defined by changing the category in which the algebras takes their values still considering the same algebraic structure on sets. For that purpose, we recall that if $\mathcal{C}$ and $\mathcal{D}$ are both fibre-complete categories of sets with structure, every functor $V : \mathcal{D} \to \mathcal{C}$ between them that preserves the underlying sets and the greatest lower bounds has an inverse image functor

$$V^* : \mathcal{L}ifts(\mathbf{T}, \mathcal{C}) \longrightarrow \mathcal{L}ifts(\mathbf{T}, \mathcal{D})$$

for any (finitary) monad on $\mathbf{Set}$. The above assumption on V amounts to the existence of a left adjoint $F \dashv V$ for it, and this condition is fulfilled by most of the abstraction functor we usually consider. For instance, to any monoid $\mathcal{M} = (M, \circ, \epsilon)$ we associate the category $\mathcal{T}(\mathcal{M})$ whose objects are the transition systems labelled with elements in that monoid. These are pairs (X, ξ) made of a set of states X and a set of transitions $\xi \subset X \times M \times X$ such that

1. $x \xrightarrow{\epsilon} x \in \xi$, for all $x \in X$ and,

2. $(x \xrightarrow{m} y \in \xi \;\text{ and }\; y \xrightarrow{n} z \in \xi) \quad \Longrightarrow \quad x \xrightarrow{m \bullet n} z \in \xi.$

A morphism $f : (X, \xi) \longrightarrow (Y, \zeta)$ between two such transition systems is a mapping $f : X \to Y$ between their sets of states such that :

$$x \xrightarrow{m} x' \in \xi \quad \Longrightarrow \quad f(x) \xrightarrow{m} f(x') \in \zeta.$$

Then to each monoid morphism $\sigma : \mathcal{M} \to \mathcal{N}$ there corresponds a functor between the associated categories of transitions systems : $\mathcal{T}_\sigma : \mathcal{T}(\mathcal{N}) \longrightarrow \mathcal{T}(\mathcal{M})$ given by

$$x \xrightarrow{m} x' \in \mathcal{T}_\sigma(\xi) \quad \Longrightarrow \quad x \xrightarrow{\sigma(m)} x' \in \xi$$

This functor has right and left adjoints $\exists_\sigma \dashv \mathcal{T}_\sigma \dashv \forall_\sigma$ given by

$$x \xrightarrow{n} x' \in \exists_\sigma(\xi) \quad \Longleftrightarrow \quad \exists m \in M \text{ s.t. } \sigma(m) = n \text{ and } \quad x \xrightarrow{m} x' \in \xi$$

and, $\quad x \xrightarrow{n} x' \in \forall_\sigma(\xi) \quad \Longleftrightarrow \quad \forall m \in M \text{ s.t. } \sigma(m) = n \text{ we have } \quad x \xrightarrow{m} x' \in \xi$

(those three functors act as identity on morphisms).
Then $V = \mathcal{T}_\sigma : \mathcal{T}(\mathcal{N}) \longrightarrow \mathcal{T}(\mathcal{M})$ and $W = \forall_\sigma : \mathcal{T}(\mathcal{M}) \longrightarrow \mathcal{T}(\mathcal{N})$ are both functor preserving the underlying sets and (since they have left adjoints) preserve the greatest lower bounds. Then we have functors that transport lifts in both direction

$$V^* : \mathcal{L}ifts(\mathbf{T}, \mathcal{T}(\mathcal{M})) \longrightarrow \mathcal{L}ifts(\mathbf{T}, \mathcal{T}(\mathcal{N}))$$

and

$$W^* : \mathcal{L}ifts(\mathbf{T}, \mathcal{T}(\mathcal{N})) \longrightarrow \mathcal{L}ifts(\mathbf{T}, \mathcal{T}(\mathcal{M}))$$

Notice that if $\mathcal{M} = A^*$ is the free monoid corresponding to a set A of actions, then a surjective monoid morphism $\sigma : A^* \to \mathcal{N}$ corresponds to an *observation criterion* as defined by Boudol [Bou85]. In the same way, if $\mathcal{C}$ is the category of transition systems we previously defined, the inclusion $I : \mathcal{C} \to A^*$ that extends a transition system by transitivity has a (left inverse) right adjoint P that retains the only transition of length 1. We then have, as well, an inverse image functor

$$P^* : \mathcal{L}ifts(\mathbf{T}, \mathcal{C}) \longrightarrow \mathcal{L}ifts(\mathbf{T}, \mathcal{T}(A^*))$$

It would be interesting if we could chacterize the category of algebras for the theory $V^*(\tilde{\mathbf{T}})$ so obtained in terms of the category of algebras for the theory $\tilde{\mathbf{T}}$ we started from and of the functor V. At first sight the following pull back diagram seems a good candidate :

Question : *Let $\mathcal{C}$ and $\mathcal{D}$ be both fibre-complete categories of sets with structure and $V : \mathcal{D} \to \mathcal{C}$ a functor preserving the underlying sets and the greatest lower bounds. Can one associate to any lift $\tilde{\mathbf{T}} \in \mathcal{L}ifts(\mathbf{T}, \mathcal{C})$ for $\mathbf{T}$ to $\mathcal{C}$, a corresponding lift $\hat{\mathbf{T}} \in \mathcal{L}ifts(\mathbf{T}, \mathcal{D})$ and a generalized monad morphism $(V, \Phi) : (\mathcal{D}, \hat{\mathbf{T}}) \longrightarrow (\mathcal{C}, \tilde{\mathbf{T}})$ such that the following diagram is a pull back ?*

$$\begin{tikzcd}
\mathcal{D}^{\hat{\mathbf{T}}} \arrow[r, "V^*_\Phi"] \arrow[d, "U^{\hat{\mathbf{T}}}"'] & \mathcal{C}^{\tilde{\mathbf{T}}} \arrow[d, "U^{\tilde{\mathbf{T}}}"] \\
\mathcal{D} \arrow[r, "V"] & \mathcal{C}
\end{tikzcd}$$

Unfortunately I have no answer to that question.

5 Conclusion

We can give some restrictions on the underlying algebraic structures we consider. Since they are intended to model programs and since those programs are defined by systems of recursive definitions, those algebraic structures must be closed under sets of recursive equations. Those equations may be expressed in the clone of operators of the considered monad. Now the clone of a finitary monad over the category of sets is none other than a Lawvere algebraic theory. This notion of algebraic structure closed under recursive definitions has been described in this context of algebraic theories [Bad89] by giving an axiomatisation for the fixed point calculi. We can very easily use this axiomatisation to obtain a *recursive closure* operation on our category of lifts.

The systematic translation of operational specifications into an equivalent denotational model we have described in this paper rely on the format of conditional rewrite rules introduced by De Simone. Though the method we give does not seems that sensible to a specific format, it need to be reconsidered if, for instance, we want to allow for negative premisses [Gro89]. Nevertheless we think that it should always exists some underlying algebraic structure (the structure of proof trees in the present case) as long as the operational specifications remain *structural.*

6 Appendix

This appendix is devoted to the proof of the few results we mention in the section 4.2. One of this was the following proposition.

Proposition 25 *If we let $\tilde{R}\xi$ be the set of transitions provable in the algebra (RX, ν_X) under the assumptions $\ulcorner\xi\urcorner$, we get a lift for $\mathbf{R}$ to the category $\mathcal{C}$ of transition systems. Moreover this is the least such lift for which each of the mappings φ_X be admissible from $(TX, \tilde{T}\xi)$ to $(RX, \tilde{R}\xi)$ (for all $\xi \in \mathcal{C}(X)$). i.e. $\tilde{\mathbf{R}}$ is the inverse image $\varphi^*(\tilde{\mathbf{T}})$ of $\tilde{\mathbf{T}}$ along φ.*

Proof : The first part of that proposition is proven is the same way as for the parallel result concerning $\tilde{\mathbf{T}}$ and corresponds to the first three items. In the last two, we verify that this lift is indeed the least such that each of the mappings φ_X is admissible.

1. *For every admissible mapping $f : (X, \xi) \to (Y, \zeta)$, Rf is admissible from $(RX, \tilde{R}\xi)$ to $(RY, \tilde{R}\zeta)$.*
   For this, consider a transition $t \xrightarrow{a} t' \in \tilde{R}\xi$, i.e.
   $$\ulcorner\xi\urcorner, \nu_X \vdash A : t \xrightarrow{a} t'$$
   due to the naturality of $\eta : I \to R$, and because Rf is a $\mathbf{T}$-algebra morphism from (RX, ν_X) into (RY, ν_Y) we know that
   $$< f^\circ, Rf >: (RX, \nu_X, \ulcorner\xi\urcorner) \longrightarrow (RY, \nu_Y, \ulcorner\zeta\urcorner)$$

is a morphism in $\mathcal{K}$ (f° is the mapping $f \times A \times f$ restricted to ξ and co-restricted to ζ). Thanks to the functoriality of $\mathcal{D}$ we deduce

$$\ulcorner \zeta \urcorner, \nu_Y \vdash A < f, Rf >: Rf(t) \xrightarrow{a} Rf(t')$$

hence $Rf(t) \xrightarrow{a} Rf(t') \in \tilde{R}\zeta$.

2. *η_X is admissible from (X, ξ) to $(RX, \tilde{R}\xi)$.*
   This comes from the fact that for every transition $x \xrightarrow{a} x' \in \xi$ one has

$$\ulcorner \xi \urcorner, \nu_X \vdash < x \xrightarrow{a} x' >: < x > \xrightarrow{a} < x' >$$

3. *μ_X is admissible from $(RRX, \tilde{R}\tilde{R}\xi)$ to $(RX, \tilde{R}\xi)$.*
   For each transition $\mathcal{T} = t \xrightarrow{a} t'$ in $\tilde{R}\xi$ we choose a proof tree $B_{\mathcal{T}} \in$ AR-P(ξ, RX) for which

$$\ulcorner \xi \urcorner, \nu_X \vdash B_{\mathcal{T}} : t \xrightarrow{a} t' \in RX \times A \times RX$$

   We notice the two following facts

   (a) the composite $\mu \circ \eta_R : R \to RR \to R$ is the identity and,

   (b) μ_X is a $\mathbf{T}$-algebra morphism from (RRX, ν_{RX}) to (RX, ν_X) :

$$\begin{array}{rcll} \mu_X \circ \nu_{RX} & = & \mu_X \circ \mu_{RX} \circ \varphi_{RRX} & \text{by definition of } \nu_{RX} \\ & = & \mu_X \circ R\mu_X \circ \varphi_{RRX} & \text{by associativity of } \mu \\ & = & \mu_X \circ \varphi_{RX} \circ T\mu_X & \text{by naturality of } \varphi \\ & = & \nu_X \circ T\mu_X & \text{by definition of } \nu_X \end{array}$$

   thanks to which we deduce that

$$< B, \mu_X >: (RRX, \nu_{RX}, \ulcorner \tilde{R}\xi \urcorner) \longrightarrow (RX, \nu_X, \mathcal{D}_{\ulcorner \xi \urcorner, \nu_X})$$

   is a morphism of the category $\mathcal{K}$. Now let a transition $u \xrightarrow{a} u' \in \tilde{R}\tilde{R}\xi$ be given, we have therefore a proof

$$\ulcorner \tilde{R}\xi \urcorner, \nu_{RX} \vdash \mathcal{A} : u \xrightarrow{a} u' \in RRX \times A \times RRX$$

   from which we can, thanks to the expansion rule, deduce the proof

$$\ulcorner \xi \urcorner, \nu_X \vdash \mathcal{A}[B, \mu_X] : \mu_X(u) \xrightarrow{a} \mu_X(u') \in RX \times A \times RX$$

   Hence $\mu_X(u) \xrightarrow{a} \mu_X(u') \in \tilde{R}\xi$.

4. *$\varphi_X : (TX, \tilde{T}\xi) \longrightarrow (RX, \tilde{R}\xi)$ is admissible*
   We notice that $< 1_\xi, \varphi_X >: (TX, \ulcorner \xi \urcorner, \mu_X^{(\mathbf{T})}) \longrightarrow (RX, \ulcorner \xi \urcorner, \nu_X)$ is a morphism of $\mathcal{K}$. This comes from the two following observations

   (a) $\varphi \circ \eta^{(\mathbf{T})} = \eta^{(\mathbf{R})}$ (notice that $\ulcorner \xi \urcorner$ does not stand for the same thing for the two objects of $\mathcal{K}$) and,

(b) φ_X is a **T**-algebra morphism from $(TX, \mu_X^{(\mathbf{T})})$ to (RX, ν_X) :

$$\begin{array}{rcll} \nu_X \circ T\varphi_X & = & \mu_X^{(\mathbf{R})} \circ \varphi_{RX} \circ T\varphi_X & \text{by definition of } \nu_X \\ & = & \varphi_X \circ \mu_X^{(\mathbf{T})} & \varphi \text{ is a monad morphism} \end{array}$$

Let $t \xrightarrow{a} t'$ be a transition of $\tilde{T}\xi$, we therefore have a proof

$$\ulcorner\xi\urcorner, \mu_X^{(\mathbf{T})} \vdash A : t \xrightarrow{a} t' \in TX \times A \times TX$$

from which (by functoriality) we get

$$\ulcorner\xi\urcorner, \nu_X \vdash A < 1_\xi, \varphi_X >: \varphi_X(t) \xrightarrow{a} \varphi_X(t') \in RX \times A \times RX$$

which means that $\varphi_X(t) \xrightarrow{a} \varphi_X(t') \in \tilde{R}\xi$.

5. *For every lift $\hat{\mathbf{R}}$ for $\mathbf{R}$ such that φ_X is admissible from $(TX, \tilde{T}\xi)$ to $(RX, \hat{R}\xi)$, we have the set theoretic inclusion $\tilde{R}\xi \subseteq \hat{R}\xi$ that holds for any set X and $\mathcal{C}$-structure $\xi \in \mathcal{C}(X)$ on that set.*
We then have to prove that $\hat{R}\xi$ contained every transition $t \xrightarrow{a} t' \in RX \times A \times RX$ for which a proof

$$\ulcorner\xi\urcorner, \nu_X \vdash A : t \xrightarrow{a} t'$$

may be found. We prove this fact by induction on the structure of the proof $A \in \mathbf{AR\text{-}P}(\xi, RX)$:

   (a) **Base case**
   The proof is of the form

$$\ulcorner\xi\urcorner, \nu_X \vdash < x \xrightarrow{a} x' >: < x > \xrightarrow{a} < x' >$$

   Since $\eta_X^{(\mathbf{R})}$ is admissible from (X, ξ) to $(RX, \hat{R}\xi)$ as composite

$$\eta_X^{(\mathbf{R})} = (X, \xi) \xrightarrow{\eta_X^{(\mathbf{T})}} (TX, \tilde{T}\xi) \xrightarrow{\varphi_X} (RX, \hat{R}\xi)$$

   of admissible mappings, we infer that the transition $< x > \xrightarrow{a} < x' >$ is in $\hat{R}\xi$.

   (b) **General case**
   The proof is of the form $A = (r : a)(A_0, \ldots, A_{n-1})$ where for every index $i \in I_r$, the proof tree A_i corresponds to a proof

$$\ulcorner\xi\urcorner, \nu_X \vdash A_i : p_i \xrightarrow{b_i} p_i'$$

   for which the condition $\phi_r(b, a)$ is fulfilled. For any other index the proof tree A_i is equal to some rational tree : $A_i = p_i \in RX$. The rational trees t and t' are then given by $t = f_{\nu_X}(p_0, \ldots, p_{n-1})$ and

$$t' = (C_r)_{\nu_X}(q_0, \ldots, q_{n-1}) \quad \text{with} \quad q_j = \begin{cases} p_j & \textbf{if} \quad j \notin I_r \\ p_j' & \textbf{if} \quad j \in I_r \end{cases}$$

By induction hypothesis we deduce $\mathcal{T}_i = p_i \xrightarrow{b_i} p_i' \in \hat{R}\xi$ for $i \in I_r$, hence the proof

$$\ulcorner \hat{R}\xi \urcorner, \mu^{(\mathbf{T})}_{RX} \vdash (r:a)(B_0,\ldots,B_{n-1}) : \mathcal{T} \xrightarrow{a} \mathcal{T}' \in TRX \times A \times TRX$$

$$\text{where} \quad B_i = \begin{cases} <\mathcal{T}_i> & \textbf{if} \quad i \notin I_r \\ \eta^{(\mathbf{T})}_{RX}(p_i) & \textbf{if} \quad i \in I_r \end{cases}$$

and where $\mathcal{T} = f[p_0,\ldots,p_{n-1}]$ [3] and $\mathcal{T}' = C_r[q_0,\ldots,q_{n-1}]$. Now, since ν_X is admissible from $(TRX, \tilde{T}\hat{R}\xi)$ to $(RX, \hat{R}\xi)$ as a composite

$$\nu_X = (TRX, \tilde{T}\hat{R}\xi) \xrightarrow{\varphi_{RX}} (RRX, \hat{R}\hat{R}\xi) \xrightarrow{\mu^{(\mathbf{R})}_X} (RX, \hat{R}\xi)$$

of admissible mappings, we get that $\nu_X(\mathcal{T}) \xrightarrow{a} \nu_X(\mathcal{T}') \in \hat{R}\xi$ that is to say $t \xrightarrow{a} t' \in \hat{R}\xi$.

□

Before proving our two remaining propositions, we state some elementary properties for the fibre-complete categories, in particular we give an equivalent definition of those categories in terms of optimal and co-optimal lifts. Let $\{f_i : X \longrightarrow (Y_i, \zeta_i)\}$ be a family (not necessarily a small set) whose elements are pairs made of an arrow $f_i : X \longrightarrow Y_i$ in the base category $\mathcal{B}$ and a $\mathcal{C}$-structure $\zeta_i \in Y_i$. The **optimal lift** of this family, if any, is the *least $\mathcal{C}$-structure ξ on X* for which each of the f_i is an admissible arrow. That is

1. the arrows f_i are $\mathcal{C}$-admissible from (X, ξ) to (Y_i, ζ_i) and,
2. if (Z, κ) is a $\mathcal{B}$-object with structure and $g : Z \to X$ an arrow for which each of the composite $f_i \circ g$ is admissible from (Z, κ) to (Y_i, ζ_i) then g is itself $\mathcal{C}$-admissible from (Z, κ) to (X, ξ).

Dually we define the **co-optimal lift** of a family $\{f_i : (X_i, \xi) \longrightarrow Y\}$. Since such an optimal (co-optimal) lift is characterized by a universal property it is *a priori* defined only up to a unique isomorphism. Nevertheless it is easy to verify, thanks to the second axiom of the categories with structure (transport of structure), that such a structure when it exists is unique. We then have the following characterization of fibre-complete categories which actually serves as a definition in [Man76].

Proposition 26 *A category of $\mathcal{B}$-objects with structure is fibre-complete if*

1. *every family $\{f_i : X \longrightarrow (Y_i, \zeta_i)\}$ has an optimal lift and,*
2. *every family $\{f_i : (X_i, \xi_i) \longrightarrow Y\}$ has a co-optimal lift.*

[3] We recall that if $g \in Tn$ et $y_0,\ldots,y_{n-1} \in Y$, the notation $g[y_0,\ldots,y_{n-1}]$ stands for the element in TY given by : $g[y_0,\ldots,y_{n-1}] = Ty(g) : 1 \xrightarrow{g} Tn \xrightarrow{Ty} TY$; and if α is a $\mathbf{T}$-algebra structure on Y, g_α is the mapping $g_\alpha : Y^n \to Y$ given by : $g_\alpha(y_0,\ldots,y_{n-1}) = (\alpha \circ Ty)(g) = \alpha(g[y_0,\ldots,y_{n-1}])$

In fact we only need to check one of the two previous requirements, the other one will be one of its corollaries (cf [Man76]).

Proof : Let $U : \mathcal{C} \to \mathcal{B}$ be a fibre-complete category of $\mathcal{B}$-objects with structure. We show that the co-optimal lift of the family $\{(X_i, \xi_i) \xrightarrow{f_i} Y \ ; \ i \in I\}$ is given by $\zeta = \bigvee_{i \in I} f_i^*(\xi_i)$.

1. f_i is admissible from (X_i, ξ_i) to (Y, ζ) as a composite of admissible arrows :
$$f_i : (X_i, \xi_i) \xrightarrow{f_i} (Y, f_i^*(\xi_i)) \xrightarrow{1_Y} (Y, \zeta)$$

2. let $g : Y \to Z$ be an arrow for which each of the composite $h_i = g \circ f_i : (X_i, \xi_i) \to (Z, \kappa)$ is admissible. we deduce the admissibility of g from $(Y, f_i^*(\xi_i))$ to (Z, κ) which is equivalent to $f_i^*(\xi_i) \leq g_*(\kappa)$, and consequently
$$\zeta = \bigvee f_i^*(\xi_i) \leq g_*(\kappa)$$

In the same way the optimal lift of the family $\{X \xrightarrow{g_i} (Y_i, \zeta_i) \ ; \ i \in I\}$ is given by $\xi = \bigwedge_{i \in I} (g_i)_*(\zeta_i)$.

Conversely, we suppose that both requirements of the proposition are fulfilled and we prove that the category is fibre-complete. First, we see that each of its fibre is a complete set with

$$\bigvee_{i \in I} \xi_i = \textbf{co-optimal lift}(\{(X, \xi_i) \xrightarrow{1_X} X \ ; \ i \in I\})$$
$$\bigwedge_{i \in I} \xi_i = \textbf{optimal lift}(\{X \xrightarrow{1_X} (X, \xi_i) \ ; \ i \in I\})$$

Moreover, any arrow $f : X \to Y$ in $\mathcal{B}$ has an inverse image and a direct image : $f^*(\xi) \in \mathcal{C}(Y)$ is the co-optimal lift of the family $\{f : (X, \xi) \to Y\}$ and $f_*(\zeta)$ the optimal lift of the family $\{f : X \to (Y, \zeta)\}$.

□

For example, if $\mathcal{C}$ is the category of transition systems, the optimal lift ξ of a family $\{X \xrightarrow{f_i} (Y_i, \zeta_i)\}$ is given by

$$x \xrightarrow{a} y \in \xi \iff \forall i \ \ f_i(x) \xrightarrow{a} f_i(y) \in \zeta_i$$

Fibre-complete categories have all types of limits and co-limits that exist in $\mathcal{B}$ and moreover they are computed at the level of the base category. More precisely, let $D : \mathcal{I} \longrightarrow \mathcal{C}$ be a diagram in $\mathcal{C}$ (with $D_i = (X_i, \xi_i)$ as components), let X be the limit of the diagram $U \circ D : \mathcal{I} \longrightarrow \mathcal{B}$ with $f_i : X \to X_i$ as limiting cone; then we get a limiting cone $f_i : (X, \xi) \longrightarrow (X_i, \xi_i)$ for the diagram D in $\mathcal{C}$ with ξ the optimal lift of the $f_i : X \longrightarrow (X_i, \xi_i)$. In the same way, a colimit in $\mathcal{C}$ is computed as the co-optimal lift of the limiting co-cone for the corresponding diagram in the base category. In particular, a fibre-complete category of sets with structure is a complete category, i.e. has limits and colimits for all set-indexed diagrams.

First we prove the following criterion for the existence of an inverse image (which by duality gives as well a criterion for the existence of a direct image).

Lemma 27 *Let $\mathcal{C}$ be a category of $\mathcal{B}$-objects with structure and $f : X \to Y$ an arrow in the base category, we say that the $\mathcal{C}$-structure is* **multiplicative** *in f when for every family $\zeta_i \in \mathcal{C}(Y)$ of $\mathcal{C}$-structures on Y and every $\mathcal{C}$-structure $\xi \in \mathcal{C}(X)$ we have*

$$\forall i \quad f : (X,\xi) \to (Y,\zeta_i) \ \text{ is admissible } \iff f : (X,\xi) \to (Y, \bigwedge_i \zeta_i) \ \text{ is admissible.}$$

Then

1. *if $\mathcal{C}$ is fibre-complete, the $\mathcal{C}$-structure is multiplicative in each of the arrow of the base category, and*

2. *if each of its fibre $(\mathcal{C}(X), \leq)$ is complete and if the $\mathcal{C}$-structure is multiplicative in $f : X \to Y$ then f has an inverse image $f^* : \mathcal{C}(X) \to \mathcal{C}(Y)$.*

Proof :
For the first part, we check on the one hand that that if f is admissible from (X,ξ) to $(Y, \bigwedge_i \zeta_i)$ then it is also admissible as a composite

$$f : (X,\xi) \xrightarrow{f} (Y, \bigwedge_i \zeta_i) \xrightarrow{1_Y} (Y,\zeta_i)$$

from (X,ξ) to (Y,ζ_i). On the other hand, if for each index i, f is admissible from (X,ξ) to (Y,ζ_i), we get $f^*(\xi) \leq \zeta_i$ hence $f^*(\xi) \leq \bigwedge_i \zeta_i$ which establishes the admissibility of f from (X,ξ) to $(Y, \bigwedge_i \zeta_i)$.
For the second part, we show that the inverse image of f is obtained as

$$f^*(\xi) = \bigwedge \{\zeta \in \mathcal{C}(Y) \ ; \ f \text{ is admissible from } (X,\xi) \text{ to } (Y,\zeta)\}$$

By the very definition we get that

$$f \text{ is admissible from } (X,\xi) \text{ to } (Y,\zeta) \implies f^*(\xi) \leq \zeta$$

for the converse part, since the $\mathcal{C}$-structure is multiplicative in f we deduce the admissibility of f from (X,ξ) to $(Y, f^*(\xi))$ hence by composition from (X,ξ) to (Y,ζ) for any $\mathcal{C}$-structure $\zeta \in \mathcal{C}(Y)$ for which $f^*(\xi) \leq \zeta$.
$\Box$.
And we now prove our two promised results

Proposition 28 *Each of the fibres of $\mathcal{L}ifts$ is complete. Moreover, if $V : \mathcal{D} \to \mathcal{C}$ is a functor that commutes with the forgetful functors and preserves the greatest lower bounds (i.e. for every family $\{\xi_i \ ; \ i \in I\}$ of $\mathcal{D}$-structures $\xi_i \in \mathcal{D}(X)$ on X one has $V(\bigwedge_{i\in I} \xi_i) = \bigwedge_{i\in I}(V\xi_i)$ [4]), then for all monad morphism $\varphi : \mathbf{T} \to \mathbf{S}$, the morphism $<\varphi, V>:<\mathbf{T}, \mathcal{C}> \longrightarrow <\mathbf{S}, \mathcal{D}>$ has an inverse image*

$$<\varphi, V>^* : \mathcal{L}ifts(\mathbf{T}, \mathcal{C}) \longrightarrow \mathcal{L}ifts(\mathbf{S}, \mathcal{D})$$

[4] the greatest lower bounds are respectively computed in $(\mathcal{D}(X), \leq)$ and $(\mathcal{C}(X), \leq)$

Proof :
For the first part we have to prove that every family $\{\tilde{\mathbf{T}}_i \ ; \ i \in I\}$ of lifts for a monad $\mathbf{T}$ to a fibre-complete category $\mathcal{C}$ of sets with structure has a greatest lower bound. We show that this greastest lower bound is in fact computed componentwise, which means that we got it by letting

$$\overline{T}\xi = \bigwedge_{i \in I} \tilde{T}_i \xi$$

for $\xi \in \mathcal{C}(X)$ (this is well-defined since $\mathcal{C}$ is fibre-complete). We only have to prove that we obtain in that way a lift for $\mathbf{T}$ to $\mathcal{C}$, it will then necessarily be the requested lower bound.

1. *For every admissible $f : (X, \xi) \to (Y, \zeta)$, Tf is admissible from $(TX, \overline{T}\xi)$ to $(TY, \overline{T}\zeta)$,*
   For each index $i \in I$ we have

   $$Tf = (TX, \overline{T}\xi) \xrightarrow{1_{TX}} (TX, \tilde{T}_i\xi) \xrightarrow{Tf} (TY, \tilde{T}_i\zeta)$$

   be admissible as a composite of admissible arrows, now this arrow may also be decomposed as

   $$Tf = (TX, \overline{T}\xi) \xrightarrow{Tf} (TY, \overline{T}\zeta) \xrightarrow{1_{TY}} (TY, \tilde{T}_i\zeta)$$

   therefore Tf is admissible from $(TX, \overline{T}\xi)$ to $(TY, \overline{T}\zeta)$.

2. *For every $\mathcal{C}$-structure $\xi \in \mathcal{C}(X)$, η_X is admissible from (X, ξ) to $(TX, \overline{T}\xi)$,*
   This simply follows from the fact that for any index $i \in I$ the following arrow is admissible

   $$\eta_X = (X, \xi) \xrightarrow{\eta_X} (TX, \overline{T}\xi) \xrightarrow{1_{TX}} (TX, \tilde{T}_i\xi)$$

3. *For every $\mathcal{C}$-structure $\xi \in \mathcal{C}(X)$, μ_X is admissible from $(TTX, \overline{TT}\xi)$ to $(TX, \overline{T}\xi)$.*

   For each index $i \in I$ we have

   $$\mu_X = (TTX, \overline{TT}\xi) \xrightarrow{1_{TTX}} (TTX, \tilde{T}_i\overline{T}\xi) \xrightarrow{T1_{TX}} (TTX, \tilde{T}_i\tilde{T}_i\xi) \xrightarrow{\mu_X} (TX, \tilde{T}_i\xi)$$

   be admissible as a composite of admissible arrows, now this arrow may also be decomposed as

   $$\mu_X = (TTX, \overline{TT}\xi) \xrightarrow{\mu_X} (TX, \overline{T}\xi) \xrightarrow{1_{TX}} (TX, \tilde{T}_i\xi)$$

   therefore μ_X is admissible from $(TTX, \overline{TT}\xi)$ to $(TX, \overline{T}\xi)$.

For the second part, it remains to prove that the structure is multiplicative in $<\varphi, V>$ when V preserves the greatest lower bounds. For that purpose, let $<\varphi, V>$ be admissible from $(b, \tilde{\mathbf{T}})$ to $(b', \hat{\mathbf{S}}_i)$ for a whole family $\{\hat{\mathbf{S}}_i \ ; \ i \in I\}$ of lifts for $\mathbf{S}$

to $\mathcal{D}$. This means that, for each index $i \in I$ and each $\mathcal{C}$-structure $\xi \in \mathcal{C}(X)$, φ_X is $\mathcal{C}$-admissible from $(TX, \tilde{T}V\xi)$ to $(SX, V\hat{S}_i\xi)$. Since $\mathcal{C}$ is fibre-complete this is equivalent to the fact that φ_X is $\mathcal{C}$-admissible from $(TX, \tilde{T}V\xi)$ to $(SX, \bigwedge_{i\in I}(V\hat{S}_i\xi))$. Now, since V preserves the greatest lower bounds and since those are computed componentwise in $\mathcal{L}ifts$ we deduce

$$\bigwedge_{i\in I}(V\hat{S}_i\xi) = V(\bigwedge_{i\in I}(\hat{S}_i\xi)) = V(\bigwedge_{i\in I}\hat{S}_i)\xi$$

hence, $<\varphi, V>$ is admissible from $(b, \tilde{\mathbf{T}})$ to $(b', \bigwedge_{i\in I}\hat{\mathbf{S}}_i)$.
□

Proposition 29 *Let* $\mathbf{T}$ *and* $\mathbf{S}$ *be two finitary monads over sets, for every lift* $\tilde{\mathbf{T}} \in \mathcal{L}ifts(\mathbf{T}, \mathcal{C})$ *for* $\mathbf{T}$ *and monad morphism* $\varphi : \mathbf{T} \to \mathbf{S}$ *we have a pull back*

$$\begin{tikzcd}
\mathcal{C}^{\tilde{\mathbf{S}}} \arrow[r, "U'^*"] \arrow[d, "\mathcal{C}^{\tilde{\varphi}}"'] & \mathbf{Set}^{\mathbf{S}} \arrow[d, "\mathbf{Set}^{\varphi}"] \\
\mathcal{C}^{\tilde{\mathbf{T}}} \arrow[r, "U^*"'] & \mathbf{Set}^{\mathbf{T}}
\end{tikzcd}$$

where

1. $\tilde{\mathbf{S}} = \varphi^*(\mathbf{S})$ *is the inverse image of* $\tilde{\mathbf{T}}$ *along* φ, *that is the least lift for* $\mathbf{S}$ *to* $\mathcal{C}$ *for which each of the component of* φ *is admissible,*

2. $\tilde{\varphi} : \tilde{\mathbf{T}} \longrightarrow \tilde{\mathbf{S}}$ *is the resulting monad morphism :*

$$\tilde{\varphi}_{(A,\xi)} = \varphi_A : (TA, \tilde{T}\xi) \longrightarrow (SA, \tilde{S}\xi) \quad \textit{and,}$$

3. U^* *et* U'^* *are the respective lifts for* $U : \mathcal{C} \to \mathbf{Set}$ *to the categories of algebras.*

Proof : The main difficulty is to prove that the given pull back does indeed correspond to a lift of the monad $\mathbf{S}$ to the category $\mathcal{C}$. It is then easy to check that the resulting lift must be the inverse image of $\tilde{\mathbf{T}}$ along φ. Consequently the proof proceeds in two stages.
1^{st} **stage :**
We recall (cf e.g. [Gro71]) that the pullback of two functors $F : \mathcal{X} \to \mathcal{C}$ and $G : \mathcal{Y} \to \mathcal{C}$ is the triple $(\mathcal{Z}, \pi_1, \pi_2)$

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{X} \arrow[r, "F"'] & \mathcal{C}
\end{tikzcd}$$

in which $\mathcal{Z}$ is the category whose objects are the pairs (X, Y) made of an object X of $\mathcal{X}$ with an object Y of $\mathcal{Y}$ such that $F(X) = G(Y)$. The morphisms $(\varphi, \psi) : (X, Y) \to (X', Y')$ in $\mathcal{Z}$ corresponds to a morphism $\varphi : X \to X'$ of $\mathcal{X}$ together with a morphism $\psi : Y \to Y'$ of $\mathcal{Y}$ for which $F(\varphi) = G(\psi)$. The two projections $\pi_1 : \mathcal{Z} \to \mathcal{X}$ and $\pi_2 : \mathcal{Z} \to \mathcal{Y}$ are then given by :

$$\pi_1 : \mathcal{Z} \longrightarrow \mathcal{X} = \begin{cases} (X, Y) & \mapsto X \\ (\varphi, \psi) & \mapsto \varphi \end{cases}$$

$$\pi_2 : \mathcal{Z} \longrightarrow \mathcal{Y} = \begin{cases} (X, Y) & \mapsto Y \\ (\varphi, \psi) & \mapsto \psi \end{cases}$$

Let $(\mathcal{X}, \pi_1, \pi_2)$ be the pull back of $U^* : \mathcal{C}^{\tilde{\mathbf{T}}} \to \mathbf{Set}^{\mathbf{T}}$ along $\mathbf{Set}^{\varphi} : \mathbf{Set}^{\mathbf{S}} \to \mathbf{Set}^{\mathbf{T}}$. The category $\mathcal{X}$ is therefore isomorphic to the category of sets with structure whose objects are triples (A, β, ξ) in which β is a $\mathbf{S}$-algebra structure on the set A and ξ is a $\mathcal{C}$-structure on A such that $\alpha = \beta \circ \varphi_A$ is $\mathcal{C}$-admissible from $(TA, \tilde{T}\xi)$ to (A, ξ); and whose arrows are $\mathbf{S}$-algebras morphisms $\psi : (A, \beta) \longrightarrow (A', \beta')$ that are $\mathcal{C}$-admissible from (A, ξ) to (A', ξ').

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathbf{Set}^{\mathbf{S}} \arrow[d, "\mathbf{Set}^{\varphi}"] \\ \mathcal{C}^{\tilde{\mathbf{T}}} \arrow[r, "U^*"'] & \mathbf{Set}^{\mathbf{T}} \end{tikzcd}$$

And according to that representation, the projections π_1 and π_2 are given by

$$\pi_1 : \mathcal{X} \longrightarrow \mathcal{C}^{\tilde{\mathbf{T}}} = \begin{cases} (A, \beta, \xi) & \mapsto (A, \beta \circ \varphi_A, \xi) \\ \psi & \mapsto \psi \end{cases}$$

$$\pi_2 : \mathcal{X} \longrightarrow \mathbf{Set}^{\mathbf{S}} = \begin{cases} (A, \beta, \xi) & \mapsto (A, \beta) \\ \psi & \mapsto \psi \end{cases}$$

We are seeking for a lift $\tilde{\mathbf{S}}$ for the monad $\mathbf{S}$ such that $\mathcal{C}^{\tilde{\mathbf{S}}}$ is isomorphic to the category $\mathcal{X}$. We recall that $\mathcal{C}^{\tilde{\mathbf{S}}}$ is isomorphic to the category of sets with structure whose objects are triples (A, β, ξ) in which β is a $\mathbf{S}$-algebra structure on the set A and ξ is a $\mathcal{C}$-structure on A such that β is $\mathcal{C}$-admissible from $(SA, \tilde{S}\xi)$ to (A, ξ); and whose arrows are the $\mathbf{S}$-algebras morphisms $\psi : (A, \beta) \longrightarrow (A', \beta')$ $\mathcal{C}$-admissible from (A, ξ) to (A', ξ').
We so obtain our required isomorphism if the $\mathcal{C}$-structure $\tilde{S}\xi$ on A is to satisfy the following universal property

> Let β a $\mathbf{S}$-algebra structure on A and $\alpha = \beta \circ \varphi_A$ the corresponding $\mathbf{T}$-algebra, then β is $\mathcal{C}$-admissible from $(SA, \tilde{S}\xi)$ to (A, ξ) if, and only if, α is $\mathcal{C}$-admissible from $(TA, \tilde{T}\xi)$ to (A, ξ).

For that reason, it seems quite natural to define $\tilde{S}\xi$ as the co-optimal lift of

$$\varphi_A : (TA, \tilde{T}\xi) \longrightarrow SA.$$

Unfortunately, this is not appropriate because it won't generally make of μ_A an admissible mapping from $(SSA, \tilde{S}(\tilde{S}(\xi)))$ to $(SA, \tilde{S}(\xi))$. For instance if $\mathcal{C}$ is the category of transition systems, and $\varphi : \mathbf{T} \to \mathbf{R}$ is the embedding of the terms in the rational trees; with the previous definition one would find in $\tilde{\mathbf{R}}\xi$ only the transitions having the form $\varphi(t) \xrightarrow{a} \varphi(t')$ for $t \xrightarrow{a} t' \in \tilde{T}\xi$. In particular there would be no transition between trees of infinite depth. On the other side, if we suppose the substitution operation on rational trees to be admissible, we have as an easy corollary that

> For every $\mathcal{C}$-structure ζ on Y and admissible arrow $v : (Y, \zeta) \to (RX, \tilde{R}\xi)$ one has
>
> $$t \xrightarrow{a} t' \in \tilde{T}\zeta \quad \Longrightarrow \quad t[v] \xrightarrow{a} t'[v] \in \tilde{R}\xi$$
>
> where $t[v] = v^*(t)$ stands for the rational tree obtained from t through the valuation v (i.e. $v^* = \mu_X^{(\mathbf{R})} \circ Rv \circ \varphi_X : TY \to RX$ is the inductive extension of $v : Y \to RX$).

and, apart from pathological cases for $\tilde{\mathbf{T}}$, this property allows us to deduce transitions between rational trees of infinite depth.

The suitable structure $\tilde{S}\xi$ is defined as follows : to each $\mathcal{C}$-object (A, ξ) we associate the family $\mathcal{F}(A, \xi)$ whose objects are the 4-tuples (B, ζ, f, β) in which B is a set, $\zeta \in \mathcal{C}(B)$ is a $\mathcal{C}$-structure sur B, f an arrow $\mathcal{C}$-admissible from (A, ξ) to (B, ζ) and $\beta : SB \to B$ is a $\mathbf{S}$-algebra structure on B such that the corresponding $\mathbf{T}$-algebra structure $\alpha = \beta \circ \varphi_B$ is $\mathcal{C}$-admissible from $(TB, \tilde{T}\zeta)$ to (B, ζ). And we let $\tilde{S}\xi$ be the optimal lift of the family

$$\{SA \xrightarrow{Sf} SB \xrightarrow{\beta} (B, \zeta)\ ;\ (B, \zeta, f, \beta) \in \mathcal{F}(A, \xi)\}$$

In the following lines we verify that we actually obtain a lift $\tilde{\mathbf{S}}$ for the monad $\mathbf{S}$, that the mappings φ_A are $\mathcal{C}$-admissible from $(TA, \tilde{T}\xi)$ to $(SA, \tilde{S}\xi)$ and that the universal property we gave above is satisfied.

1. *φ_A is $\mathcal{C}$-admissible from $(TA, \tilde{T}\xi)$ to $(SA, \tilde{S}\xi)$*
   Let (B, f, β, ζ) be an element of $\mathcal{F}(A, \xi)$, $\alpha = \beta \circ \varphi_B$ is therefore $\mathcal{C}$-admissible from $(TB, \tilde{T}\zeta)$ to (B, ζ) and Tf is $\mathcal{C}$-admissible from $(TA, \tilde{T}\xi)$ to $(TB, \tilde{T}\zeta)$, hence $\beta \circ Sf \circ \varphi_A = \beta \circ \varphi_B \circ Tf$ is $\mathcal{C}$-admissible from $(TA, \tilde{T}\xi)$ to (B, ζ) and φ_A is $\mathcal{C}$-admissible from $(TA, \tilde{T}\xi)$ to $(SA, \tilde{S}\xi)$.

2. *If β is a $\mathbf{S}$-algebra structure on A and $\alpha = \beta \circ \varphi_A$ the corresponding $\mathbf{T}$-algebra structure, then β is $\mathcal{C}$-admissible from $(SA, \tilde{S}\xi)$ to (A, ξ) if, and only if, α is $\mathcal{C}$-admissible from $(TA, \tilde{T}\xi)$ to (A, ξ).*
   Actually, if β is $\mathcal{C}$-admissible, so is α as a composite of two admissible arrows. Conversely, if α is $\mathcal{C}$-admissible then $(A, \xi, 1_A, \beta)$ is an element of $\mathcal{F}(A, \xi)$ and therefore $\beta \circ S1_A = \beta$ is $\mathcal{C}$-admissible.

3. *If f is a $\mathcal{C}$-admissible arrow from (A,ξ) to (B,ζ), then Sf is a $\mathcal{C}$-admissible arrow from $(SA,\tilde{S}\xi)$ to $(SB,\tilde{S}\zeta)$.*
Let (C,κ,g,β) be an element of $\mathcal{F}(B,\zeta)$, then $(C,\kappa,g\circ f,\beta)$ is an element of $\mathcal{F}(A,\xi)$ and therefore $\beta\circ Sg\circ Sf$ is a $\mathcal{C}$-admissible arrow from $(SA,\tilde{S}\xi)$ to (C,κ). Thanks to the definition of $\tilde{S}\zeta$ we deduce that Sf is $\mathcal{C}$-admissible from $(SA,\tilde{S}\xi)$ to $(SB,\tilde{S}\zeta)$.

4. *For every $\mathcal{C}$-object (A,ξ), η_A is $\mathcal{C}$-admissible from (A,ξ) to $(SA,\tilde{S}\xi)$.*
Let $(B,\zeta,f,\beta)\in\mathcal{F}(A,\xi)$, $(\beta\circ Sf)\circ\eta_A=\beta\circ\eta_B\circ f=f$ is $\mathcal{C}$-admissible from (A,ξ) to (B,ζ) hence the result.

5. *For every $\mathcal{C}$-object (A,ξ), μ_A is $\mathcal{C}$-admissible from $(SSA,\tilde{S}\tilde{S}\xi)$ to $(SA,\tilde{S}\xi)$.*
Let $(B,\zeta,f,\beta)\in\mathcal{F}(A,\xi)$, according to the item (2) the $\mathbf{S}$-algebra structure β is $\mathcal{C}$-admissible from $(SB,\tilde{S}\zeta)$ to (B,ζ) and according to (3) $S\beta$ is $\mathcal{C}$-admissible from $(SSB,\tilde{S}\tilde{S}\zeta)$ to $(SB,\tilde{S}\zeta)$ and SSf from $(SSA,\tilde{S}\tilde{S}\xi)$ to $(SSB,\tilde{S}\tilde{S}\zeta)$. The composite $\beta\circ S\beta\circ SSf$ is therefore $\mathcal{C}$-admissible from $(SSA,\tilde{S}\tilde{S}\xi)$ to (B,ζ). But this composite is none other than $\beta\circ Sf\circ\mu_A$, hence the desired result.

Now, via the isomorphism $\mathcal{C}^{\tilde{\mathbf{S}}}\cong\mathcal{X}$ the projections π_1 and π_2 correspond respectively to $\mathcal{C}^{\tilde{\varphi}}$ and U'^{*} where $\tilde{\varphi}:\tilde{\mathbf{T}}\longrightarrow\tilde{\mathbf{S}}$ is the monad morphism whose components are those of φ :

$$\tilde{\varphi}_{(A,\xi)}=\varphi_A:(TA,\tilde{T}\xi)\longrightarrow(SA,\tilde{S}\xi)$$

(φ_A is admissible from $(TA,\tilde{T}\xi)$ to $(SA,\tilde{S}\xi)$ and it clearly is a monad morphism since φ is such) and where U'^{*} is the lift for U corresponding to $\tilde{\mathbf{S}}$.
2^{nd} **stage :**
We prove in that second stage that the so obtained lift of $\mathbf{S}$ is the inverse image $\varphi^{*}(\tilde{\mathbf{T}})$ of $\tilde{\mathbf{T}}$ along φ. We have defined an order relation on the set of lifts for $\mathbf{S}$ to $\mathcal{C}$ by :

$$\check{\mathbf{S}}\sqsubseteq\hat{\mathbf{S}}\quad\Longleftrightarrow\quad\forall X\text{ and }\xi\in\mathcal{C}(X)\quad\check{S}\xi\leq\hat{S}\xi$$

$\check{\mathbf{S}}\sqsubseteq\hat{\mathbf{S}}$ is equivalent to the fact that the identity $1_{\mathbf{S}}:\check{\mathbf{S}}\to\hat{\mathbf{S}}$ is a monad morphism (in $\mathcal{M}on(\mathcal{C})$). $\varphi^{*}(\tilde{\mathbf{T}})$ is the least lift of $\mathbf{S}$, as regard this order relation, for which each of the arrows

$$\varphi_X:(TX,\tilde{T}\xi)\longrightarrow(SX,\tilde{S}\xi)$$

are admissible. We deduce, in particular the following monad morphism

$$1_{\mathbf{S}}:\varphi^{*}(\tilde{\mathbf{T}})\longrightarrow\tilde{\mathbf{S}}\tag{2}$$

with $\tilde{\mathbf{S}}$ the lift we defined in the first stage of this proof.

Let $\mathcal{L}_\varphi(\mathbf{S},\mathcal{C})$ be the set of lifts $\hat{\mathbf{S}}$ for $\mathbf{S}$ to the category $\mathcal{C}$ that makes each of the φ-component an admissible arrow. If $\hat{\mathbf{S}}\in\mathcal{L}_\varphi(\mathbf{S},\mathcal{C})$ we obtain a monad morphism $\hat{\varphi}:\tilde{\mathbf{T}}\to\hat{\mathbf{S}}$ by letting $\hat{\varphi}_{(X,\xi)}=\varphi_X$ for each $\mathcal{C}$-structure $\xi\in\mathcal{C}(X)$. Moreover the following diagram commutes

$$\begin{tikzcd}
\mathcal{C}^{\hat{\mathbf{S}}} \arrow[r, "U'^*"] \arrow[d, "\mathcal{C}^{\hat{\varphi}}"'] & \mathbf{Set}^{\mathbf{S}} \arrow[d, "\mathbf{Set}^{\varphi}"] \\
\mathcal{C}^{\tilde{\mathbf{T}}} \arrow[r, "U^*"'] & \mathbf{Set}^{\mathbf{T}}
\end{tikzcd}$$

The lift $\tilde{\varphi} : \tilde{\mathbf{T}} \rightarrow \tilde{\mathbf{S}}$ for φ described in the first stage is the *least* in the sense that for every $\hat{\mathbf{S}} \in \mathcal{L}_{\varphi}(\mathbf{S}, \mathcal{C})$, the corresponding monad morphism $\hat{\varphi} : \tilde{\mathbf{T}} \rightarrow \hat{\mathbf{S}}$ can be decomposed **in a unique way** as $\hat{\varphi} = \hat{\alpha} \circ \tilde{\varphi}$ where $\hat{\alpha} : \tilde{\mathbf{S}} \rightarrow \hat{\mathbf{S}}$ a morphism in $\mathcal{M}on(\mathcal{C})$. In particular, there exists a unique monad morphism

$$\alpha : \tilde{\mathbf{S}} \longrightarrow \varphi^*(\tilde{\mathbf{T}}) \tag{3}$$

such that $\hat{\varphi} = \alpha \circ \tilde{\varphi}$.

Thanks to both monad morphisms (2) and (3) we get

$$\beta = \tilde{\mathbf{S}} \xrightarrow{\alpha} \varphi^*(\tilde{\mathbf{T}}) \xrightarrow{1_{\mathbf{S}}} \tilde{\mathbf{S}}$$

for which $\tilde{\varphi} = \beta \circ \tilde{\varphi}$, we deduce that the components of β (that are as the same time the components of α) are equals to the identities. That means that 1_{SX} is admissible from $(SX, \tilde{\mathbf{S}}\xi)$ to $(SX, \varphi^*(\tilde{\mathbf{T}})\xi)$ for every $\xi \in \mathcal{C}(X)$, i.e. $\tilde{\mathbf{S}} \sqsubseteq \varphi^*(\tilde{\mathbf{T}})$. Hence $\tilde{\mathbf{S}} = \varphi^*(\tilde{\mathbf{T}})$
$\square$

References

[Bad89] Eric Badouel. Algebraically closed theories. *Proceeding MFCS'89* , Lecture Notes in Computer Science vol 379, 1989.

[Ben75] David B. Benson. Algebraic structures in categories of derivations. *Information and Control*, 28, 1975.

[Bou85] Gérard Boudol. Notes on algebraic calculi of processes. *in Logics and models of concurrent systems (K Apt, Ed) Nato Asi Series F13*, 1985.

[BW85] M. Barr and C. Wells. *Toposes, triples and theories.* Springer Verlag, 1985.

[dS84] Robert de Simone. *Calculabilité et expressivité dans l'algèbre de processus MEIJE.* Thèse de 3eme Cycle, Paris VII, 1984.

[Eyt80] Michel Eytan. Grothendieck topologies on formal grammars. 3^{rd} *workshop on algebraic and categorical methods in computer science, Herdecke-Dortmund*, 1980.

[Gro71] Alexandre Grothendieck. Catégories fibrées et descente. *in* Revêtement étales et groupe fondamental, *exposé VI du séminaire de géométrie algébrique du bois Marie 1960/61 (SGA 1)*, Lecture Notes in Mathematics vol 224, 1971.

[Gro89] J.F. Groote. Transition system specifications with negative premises. *Report CS-R8950, CWI, Amsterdam*, 1989.

[Lam69] Joachim Lambek. Deductive systems ans categories II. *Category Theory, homology Theory and their Applications I. Springer Verlag Lecture Notes in Mathematics*, 86, 1969.

[Lan71] Saunders Mac Lane. *Categories for the working mathematician.* Springer Verlag, GTM 5, 1971.

[Man76] E. G. Manes. *Algebraic theories.* Springer Verlag, 1976.

[Mes90] José Meseguer. Rewriting as a unified model of concurrency. *Workshop on Concurrency and Compositionality, Pisa*, 1990.

[Mil80] Robin Milner. *A calculus of communicating systems.* Lecture Notes in Computer Science vol 92, 1980.

[Plo81] Gordon Plotkin. *A structural approach to operational semantics.* Rep Daimi FN 19, Aarhus Univ., 1981.

[Rao84] Jean-Claude Raoult. On graph rewriting. *Theoretical Computer Science*, 32, 1984.

[RS87] D.E. Rydeheard and J.G Stell. Foundations of equational deduction : a categorical treatment of equational proofs and unification algorithms. in *Category theory and computer science* Lecture Notes in Computer Science vol 283, 1987.

[Str72] Ross Street. The formal theory of monads. *Journal of pure and applied algebra*, 2, 1972.

[Sza78] M.E. Szabo. *Algebra of proofs.* North Holland, Amsterdam, 1978.

LISTE DES DERNIERES PUBLICATIONS INTERNES PARUES A L'IRISA

PI 518 MULTISCALE SYSTEM THEORY
Albert BENVENISTE, Ramine NIKOUKHAH, Alan S. WILLSKY
Février 1990, 30 Pages.

PI 519 PANDORE : A SYSTEM TO MANAGE DATA DISTRIBUTION
Françoise ANDRE, Jean-Louis PAZAT, Henry THOMAS
Février 1990, 14 Pages.

PI 520 SCHEDULING AFFINE PARAMETERIZED RECURRENCES BY MEANS OF VARIABLE DEPENDENT TIMING FUNCTIONS
Christophe MAURAS, Patrice QUINTON,
Sanjay RAJOPADHYE, Yannick SAOUTER
Février 1990, 14 Pages.

PI 521 COMPUTABILITY OF RECURRENCE EQUATIONS
Yannick SAOUTER, Patrice QUINTON
Février 1990, 28 Pages.

PI 522 PROGRAMMING BY MULTISET TRANSFORMATION
Jean-Pierre BANATRE, Daniel LE METAYER
Mars 1990, 26 Pages.

PI 523 GOTHIC MEMORY MANAGEMENT : A MULTIPROCESSOR SHARED SINGLE LEVEL STORE
Béatrice MICHEL
Mars 1990, 20 Pages.

PI 524 ORDER NOTIONS AND ATOMIC MULTICAST IN DISTRIBUTED SYSTEMS : A SHORT SURVEY
Michel RAYNAL
Mars 1990, 18 Pages.

PI 525 MULTI-SCALE AUTOREGRESSIVE PROCESSES
Michèle BASSEVILLE, Albert BENVENISTE
Mars 1990, 136 Pages.

PI 526 TRANSFORMATIONS PYRAMIDALES D'IMAGES NUMERIQUES
Nadia BAAZIZ, Claude LABIT
Mars 1990, 102 Pages.

PI 527 LE LANGAGE SIGNAL : UN EXEMPLE EN SEGMENTATION AUTOMATIQUE DE LA PAROLE CONTINUE
Claude LE MAIRE
Mars 1990, 112 Pages.

PI 528 CONDITIONAL REWRITE RULES AS AN ALGEBRAIC SEMANTICS OF PROCESSES
Eric BADOUEL
Mars 1990, 46 Pages.

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique.

ISSN 0249-6399